Problem Set 2

Problems to Theoretical Astrophysics, SS 2014

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Tutorial on Tuesday, 06.05.2014, 10.15 am

1. (Laplace-)Runge-Lenz vector

a) The absolute value of the Runge-Lenz vector $\boldsymbol{A} \equiv \boldsymbol{p} \times \boldsymbol{L} - mk\hat{\boldsymbol{r}}$ is given by

$$A^2 = m^2k^2 + 2mEL^2.$$

Derive this equation.

b) What is the meaning of $|A|$? In which direction does $\boldsymbol{A}$ point?

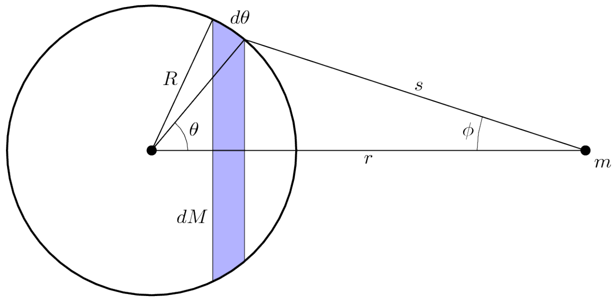
2. Planetary orbits

For the Kepler problem, determine the closed orbit with minimal energy. Compute energy and eccentricity of this orbit. Discuss the observation that the orbits of large planets in the solar system show small eccentricity, whereas for many extrasolar planets high eccentricities are found.

3. Newton's shell theorem

One of the main reasons for Newton to introduce integrals was to show that in his theory of gravitation, spherically symmetric mass distributions (i.e. $\rho(\boldsymbol{x}) = \rho(r)$; e.g. planets, stars...) could be treated as point masses. This is the main point of Newton's shell theorem.

a) Assume a thin spherical shell with radius R, surface $A(R)$, and surface density $\sigma(R) = \frac{M(R)}{A(R)}$. Compute the gravitational potential $\Phi(r)$ at a point with radius r outside of the shell by integrating the contributions of all spherical rings (see Figure).



To this end, compute first the mass $\mathrm{d}M$ of a spherical ring and its gravitational potential $\mathrm{d}\Phi(r)$.

Hint: The law of cosines yields $\cos\theta = (r^2 + R^2 - s^2)/(2rR)$, and thus $\sin\theta \mathrm{d}\theta = s\mathrm{d}s/(rR)$.

b) Compute the gravitational potential at a point inside the shell.

c) Compute the gravitational field $g(r)$ and interpret the results.

d) Assume now a spherically symmetric, but otherwise arbitrary mass distribution $\rho(\boldsymbol{x}) = \rho(r)$. Compute the gravitational potential $\Phi(r)$.

4. Lagrangian point L_1 in the Earth-Moon-system

In this exercise, we compute the position of the Lagrangian point L_1 between Earth and Moon, neglecting interactions with other objects in the solar system. As a simplification, we assume Earth (mass M) and Moon (mass m) to revolve on circular orbits around their center of mass.

a) Show that the period T of this orbit is given by

$$T^2 = \frac{4\pi^2 a^3}{G(m+M)},$$

where a denotes the distance between Earth and Moon.

b) Compute the position of the Lagrangian point L_1 as a function of M, m, and a.

Hint: This leads to a fifth-degree equation without a simple analytical solution. Find this equation and solve it numerically using a computer.