Problem Set 4

Problems to Theoretical Astrophysics, SS 2014

Prof. Dr. Friedrich Röpke, Sebastian Ohlmann
Offices: Campus Hubland Nord, 31.01.017, 31.01.003

Tutorial on Tuesday, 20.05.2014, 10.15 am

1. Adiabatic exponents

a) Show that

$$\frac{\Gamma_3 - 1}{\Gamma_1} = \frac{\Gamma_2 - 1}{\Gamma_2} = \nabla_{\mathrm{ad}}.$$

b) Determine C_V, χ_T, χ_ρ, C_P, γ, Γ_1, Γ_2, and Γ_3 for an ideal gas. Assume as equation of state $E = \frac{3}{2} N k_{\mathrm{B}} T$ and $PV = N k_{\mathrm{B}} T$.

c) Show the following relations for Γ_1, Γ_2, and Γ_3:

$$\Gamma_1 = \chi_\rho \gamma$$
$$\frac{\Gamma_2}{\Gamma_2 - 1} = \frac{T C_P}{PV} \frac{\chi_\rho}{\chi_T}$$
$$\Gamma_3 - 1 = \frac{PV}{T} \frac{\chi_T}{C_V}$$

Hint: You can use the following formulae:

$$\gamma = \frac{C_p}{C_v}$$
$$\chi_\rho = \left(\frac{\partial \ln P}{\partial \ln \rho}\right)_T$$
$$\chi_T = \left(\frac{\partial \ln P}{\partial \ln T}\right)_\rho$$
$$\Gamma_1 = \left(\frac{\partial \ln P}{\partial \ln \rho}\right)_S$$
$$\frac{\Gamma_2}{\Gamma_2 - 1} = \frac{1}{\nabla_{\mathrm{ad}}} = \left(\frac{\partial \ln P}{\partial \ln T}\right)_S$$
$$\Gamma_3 - 1 = \left(\frac{\partial \ln T}{\partial \ln \rho}\right)_S$$
$$\left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y = -1$$

2. Schwarzschild criterion

Show that an alternate criterion for *convective stability* is given by the Schwarzschild criterion,

$$\frac{\mathrm{d}S}{\mathrm{d}z} > 0.$$

3. Transition to continuum and Fermi energy

a) Assume a single particle in a box of length L. Then the momenta are quantized according to the Schrdinger equation as $\vec{p} = \frac{\hbar\pi}{L}\vec{n}$, where $\vec{n}$ are the quantum number for each dimension. In the limit of large lengths L, discrete sums over energy levels can be transformed into integrals over the continuous momentum,

$$\frac{1}{V}\sum_{\vec{n},\sigma} f(\epsilon_{\vec{n}}) \to g_s \int_0^\infty \mathrm{d}p\, f(\epsilon(p))\nu(p),$$

where $V = L^3$ is the volume, σ is the spin, and g_s the spin degeneracy.

Hint: Include an additional factor $(\Delta n)^3$, where $\Delta n = 1$, and transform it to $(\Delta p)^3$. For large L, this renders the infinitesimal quantity d^3p. Take into account $\vec{n} \geq 0$.

b) For an ideal Fermi gas at $T = 0$, all states are occupied up to the Fermi energy ϵ_F because of the Pauli principle. The number of states equals the number of particles and is given by $N = \sum_{\vec{n},\sigma} n_\mathrm{F}(\epsilon)$, where $n_\mathrm{F}(\epsilon)$ is the Fermi function.

Compute the Fermi momentum p_F and the Fermi energy ϵ_F by assuming a step function for the Fermi function.

Hint: The final result is

$$\epsilon_\mathrm{F} = \frac{\hbar^2}{2m}\left(\frac{6\pi^2 n}{g_s}\right)^{\frac{2}{3}},$$

where $n = N/V$ denotes the number density.

c) Compute the Fermi energy and the Fermi temperature $T_\mathrm{F} = \epsilon_\mathrm{F}/k_\mathrm{B}$ for nuclear matter, a metal, and a white dwarf.

Hint: For nuclear matter, consider a nucleus obeying the relation $R = 1.25 \times 10^{-15}\,\mathrm{m}\, A^{1/3}$, where A is the total number of nucleons. For the other cases, the density is given by the baryonic density with $n_\mathrm{el} = \rho\frac{Y_\mathrm{e}}{m_\mathrm{b}}$, where Y_e denotes the fraction of electrons to baryons and $m_\mathrm{b} \approx m_\mathrm{u}$ the mean baryonic mass. You can use the following values:

Iron	$\rho = 7.87\frac{\mathrm{g}}{\mathrm{cm}^3}$	$Y_\mathrm{e} = 0.466$
White dwarf	$\rho = 3 \times 10^6\frac{\mathrm{g}}{\mathrm{cm}^3}$	$Y_\mathrm{e} = 0.5$

4. Degeneracy in stars

A criterion for degeneracy in the interior of stars can be obtained by comparing the Fermi energy ϵ_F to the mean thermal energy $\frac{3}{2}k_\mathrm{B}T$.

a) Obtain a criterion for $T/\rho^{2/3}$ allowing a statement concerning the degree of degeneracy.

b) Compare the Sun and the white dwarf Sirius B with regard to the degeneracy of electrons in the interior.

   Hint: You can use the following values:

Sun	$T_c = 1.58 \times 10^7 \mathrm{K}$	$\rho_c = 162 \frac{\mathrm{g}}{\mathrm{cm}^3}$
Sirius B	$T_c = 7.6 \times 10^7 \mathrm{K}$	$\rho_c = 3 \times 10^6 \frac{\mathrm{g}}{\mathrm{cm}^3}$