Introduction to numerical methods to hyperbolic PDE's

Literature:

Randy LeVeque: Computational Methods for Astrophysical Fluid Flow, Sass Fee lecture notes, Springer Verlag, 1998

download this from: http://www.mat.univie.ac.at/~obertsch/literatur/conservation_laws.pdf

Lecture 1: Basic concepts

Outline of the first lecture

- Finite difference vs. finite volume methods
- Convergence
- Local truncation error, consistency
- Stability and the CFL condition
- Upwind methods
- Lax-Friedrichs, Lax-Wendroff
- Diffusion, dispersion, modified equations

Here is a typical example of a system of PDEs we want to numerically discretise:

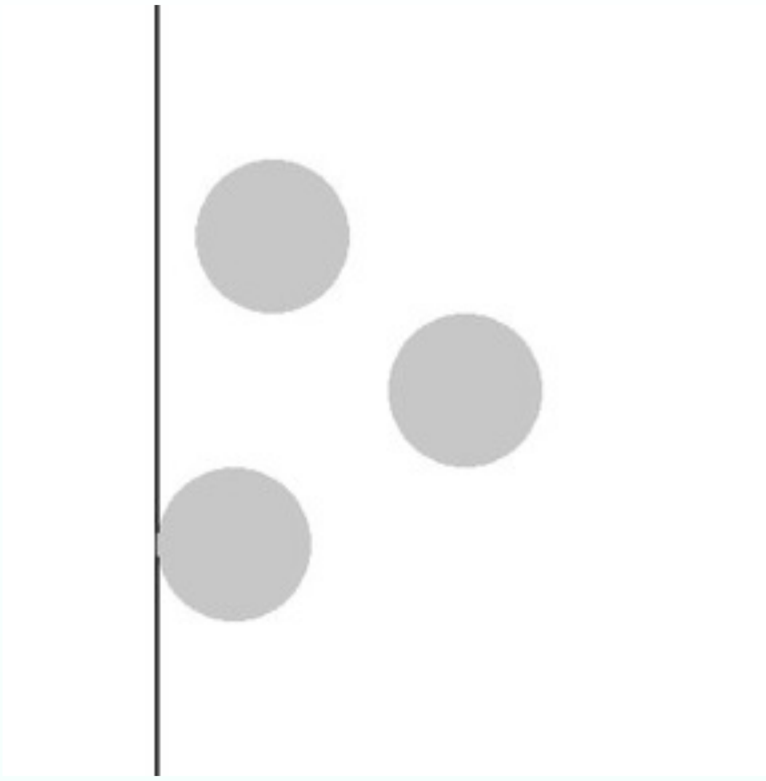
Euler equations of compressible gas dynamics:

$$\rho_t + (\rho u)_x = 0$$ conservation of mass

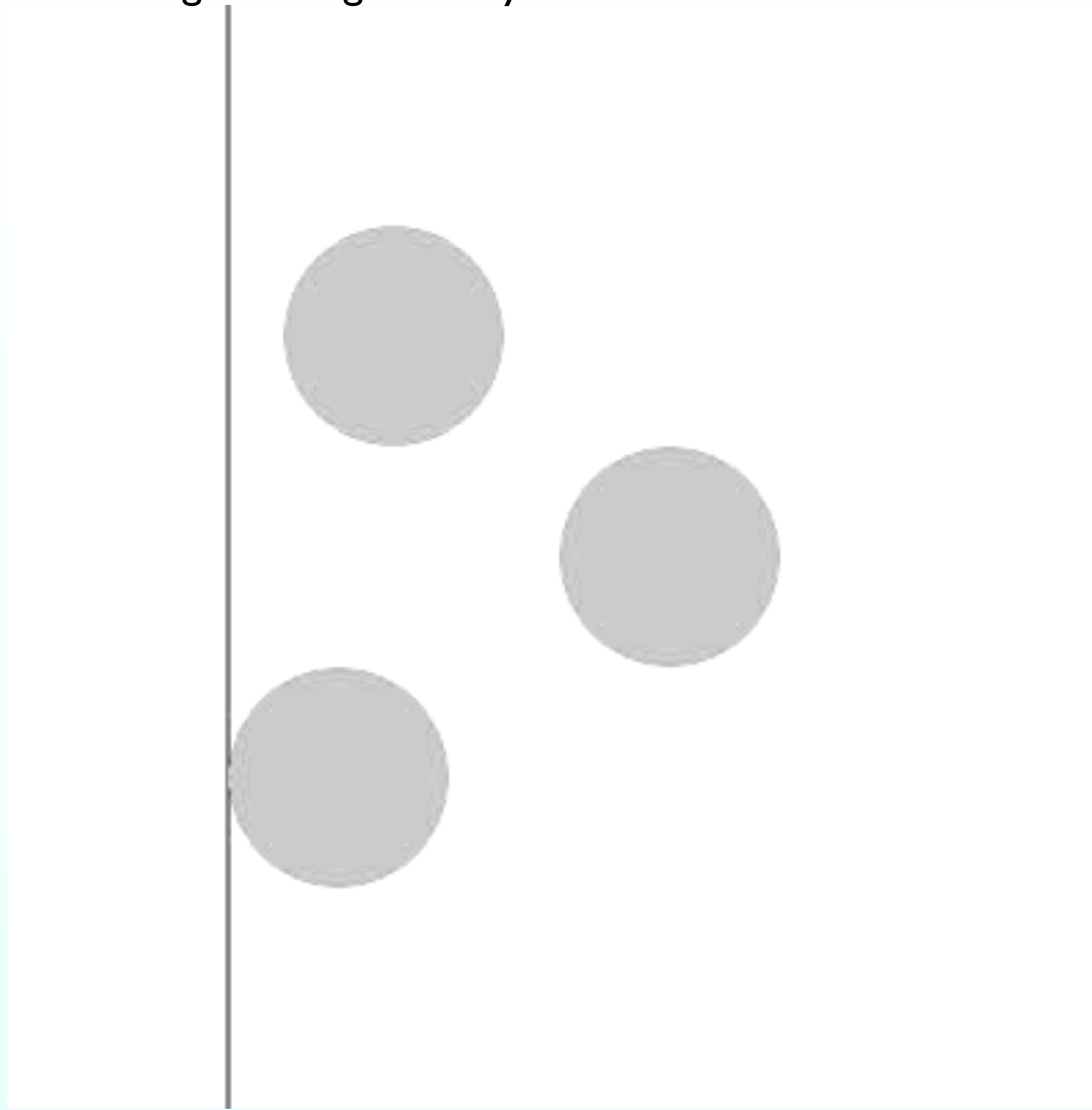
$$(\rho u)_t + (\rho u^2 + p)_x = 0$$ conservation of momentum

$$E_t + \big(u(E+p)\big)_x = 0$$ conservation of total energy

closure relationship - equation of state: $$E = \frac{p}{\gamma - 1} + \frac{1}{2}\rho u^2$$ *polytropic gas* a planar shock wave in a gas hitting three cylinders



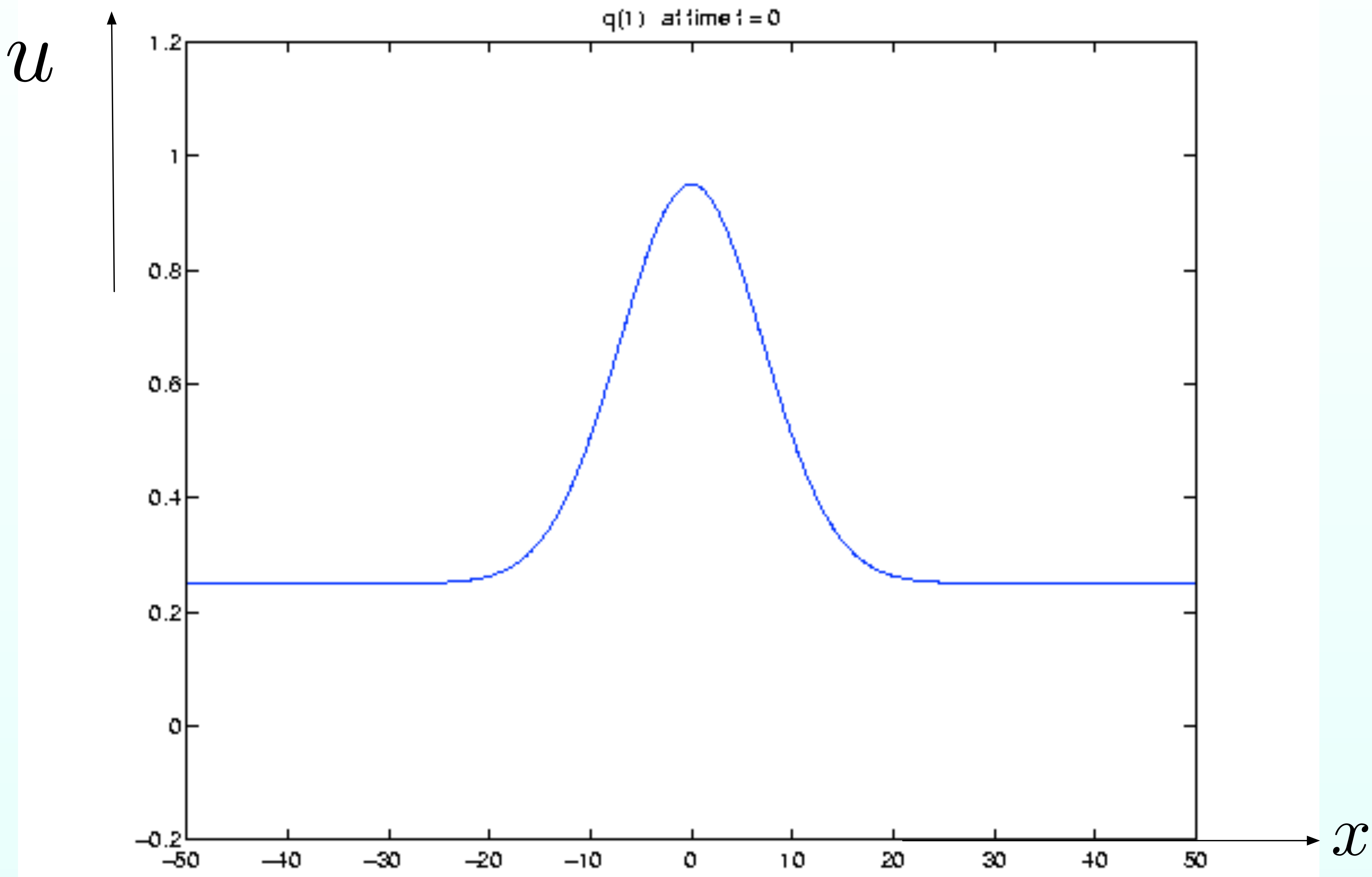
t = 0



In order to study the numerical discretization of such equations we first study simpler equations.

Given smooth initial data for such equations, the solution will evolve into something not smooth.

consider $u = u(x,t) \in R$ Burgers equation: $u_t + \left(\frac{u^2}{2}\right)_x = 0$

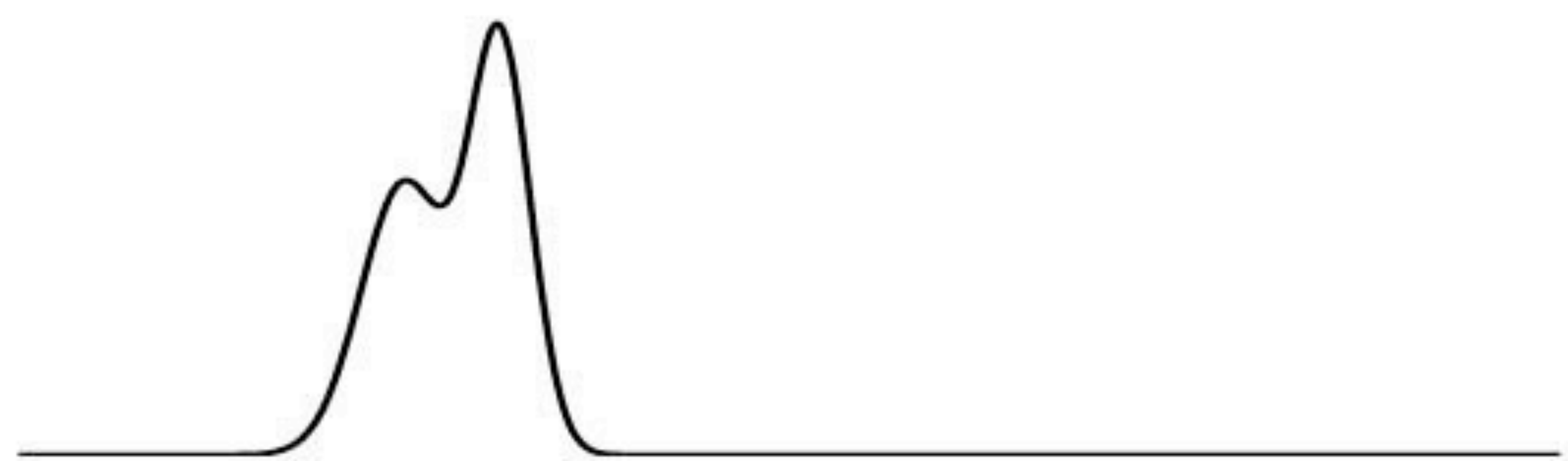


Linear advection equation

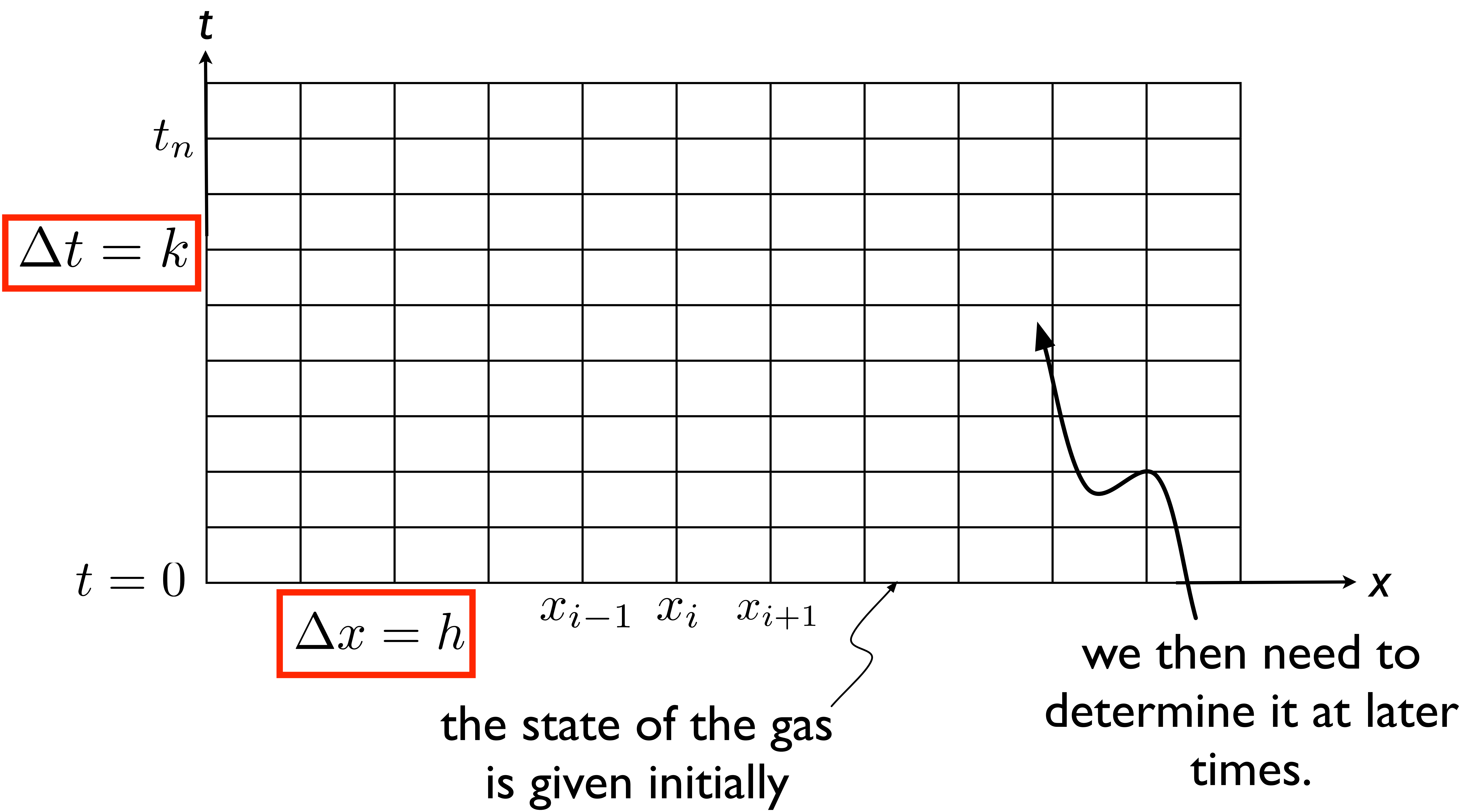
$$q_t + uq_x = 0$$

True solution: $q(x,t) = q(x - ut, 0)$

Assume $u > 0$ so flow is to the right.



Numerical methods use space- and time discretization:



Finite difference method

Based on point-wise approximations:

$$Q_i^n \approx q(x_i, t_n), \quad \text{with } x_i = ih, \;\; t_n = nk.$$

Approximate derivatives by finite differences.

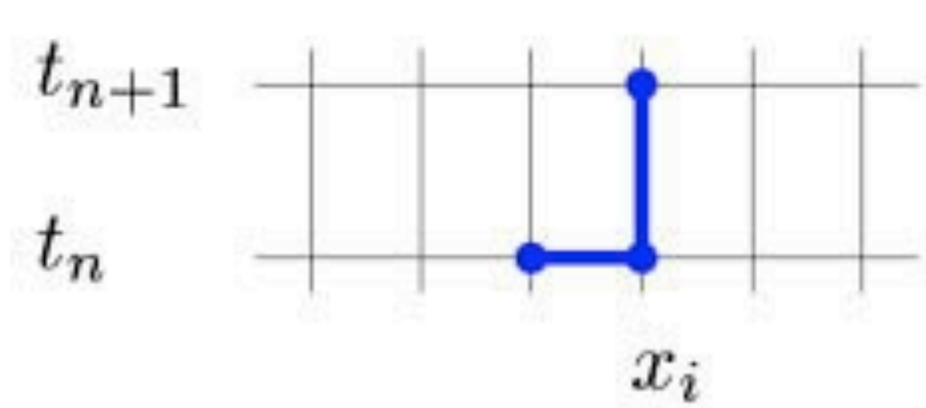
Ex: Upwind methods for advection equation $q_t + uq_x = 0$:

$$\frac{Q_i^{n+1} - Q_i^n}{k} + u\left(\frac{Q_i^n - Q_{i-1}^n}{h}\right) = 0$$

or

$$Q_i^{n+1} = Q_i^n - \frac{k}{h}u(Q_i^n - Q_{i-1}^n).$$

Stencil:



Finite volume method

Based on cell averages:

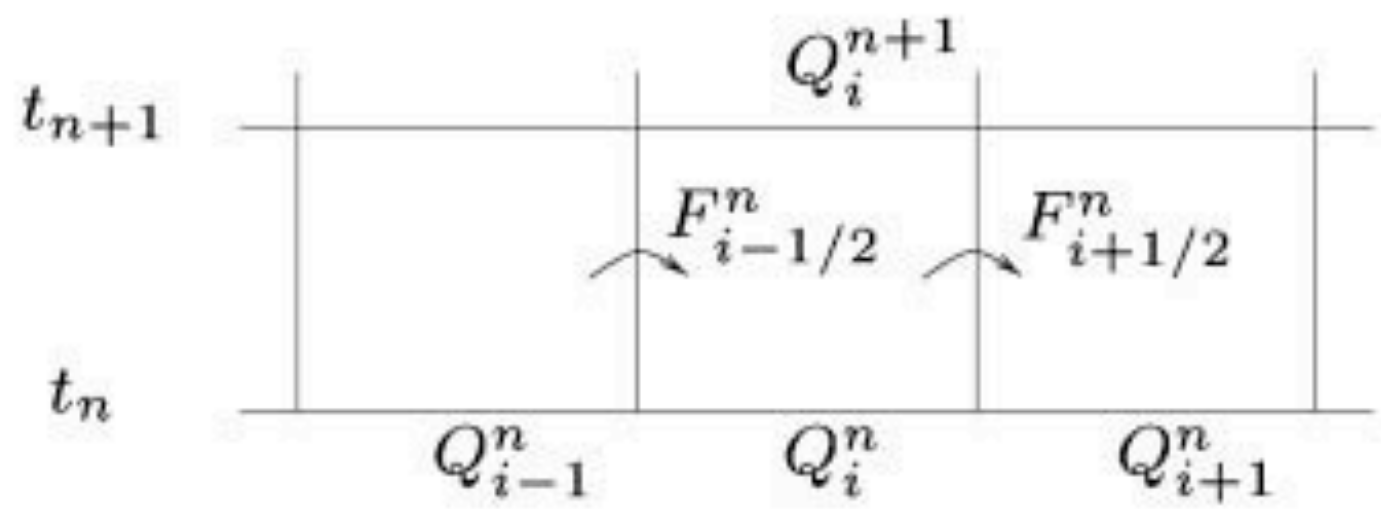
$$Q_i^n \approx \frac{1}{h} \int_{x_{i-1/2}}^{x_{i+1/2}} q(x, t_n)\, dx$$

Update cell average by flux into and out of cell:

Ex: Upwind methods for advection equation $q_t + uq_x = 0$:

$$\begin{aligned} Q_i^{n+1} &= Q_i^n - \frac{k(uQ_{i-1}^n - uQ_i^n)}{h} \\ &= Q_i^n - \frac{ku}{h}(Q_i^n - Q_{i-1}^n) \end{aligned}$$

Stencil:



Convergence

Global error $E_i^n = Q_i^n - q(x_i, t_n)$

We want: E at fixed (x, t) to approach 0 as $k,\ h \to 0$.

or $\|Q^N - q(\cdot, T)\| \to 0$ as $k,\ h \to 0$ with $Nk = T$.

Method is **order** p (globally) if

$$E(k, h) = O(k^p + h^p) \quad \text{as } k,\ h \to 0.$$

Hard to deal with directly: more and more points as grid is refined

Study **local error** and **stability**.

Local truncation error τ

Difference formula:

$$\frac{Q_i^{n+1} - Q_i^n}{k} + u\left(\frac{Q_i^n - Q_{i-1}^n}{h}\right) = 0$$

Insert true solution into formula to determine LTE:

$$\tau(x,t) = \frac{q(x,t+k) - q(x,t)}{k} + u\left(\frac{q(x,t) - q(x-h,t)}{h}\right)$$

For smooth q we can use Taylor series to expand:

$$\begin{aligned} \tau(x,t) &= (q_t + kq_{tt} + \cdots) + u\,(q_x + hq_{xx} + \cdots) \\ &= (q_t + uq_x) + kq_{tt} + hq_{xx} + \cdots \\ &= O(k+h) \end{aligned}$$

Upwind is first order accurate (locally)

Consistency

A method is **consistent** if $\tau \to 0$ as $k,\ h \to 0$.

The **one-step error** is $k\tau$:

$$k\tau = q(x,t+k) - \left(q(x,t) - \frac{uk}{h}(q(x,t) - q(x-h,t)) \right).$$

An error of this magnitude is made in each of T/k time steps.

This suggests $E \approx (T/k)(k\tau) = T\tau$:

If $\tau = O(h^p + k^p) \implies$ global error is $O(h^p + k^p)$

The method is pth order accurate

But:

This is valid **provided** the method is **stable**!

Consistency $+$ stablity $=$ convergence

Fundamental therom:

Consistency + Stability = Convergence

ODE: zero-stability, stability on $q'(t) = 0$ is enough.
Dahlquist Theorem.

Linear PDE: Lax-Richtmyer stability
Uniform power boundedness of a family of matrices
Lax equivalence Theorem.

Scalar conservation law: total variation stability

Systems of conservation laws: ?? — few convergence proofs

Lax-Richtmyer stabilty

Linear method: $Q^{n+1} = B_k Q^n$ (with k/h fixed).

The method is Lax-Richtmyer stable in some norm $\|\cdot\|$ if, for every time $T > 0$, there is a constant C_T such that

$$\|B_k^N\| \le C_T$$

for all $k,\ N$ with $Nk \le T$.

It is sufficient to show that there is an α for which

$$\|Q^{n+1}\| \le (1 + \alpha k)\|Q^n\|.$$

since then

$$\|Q^N\| \le (1 + \alpha k)^N \|Q^0\| \le e^{\alpha K N}\|Q^0\| \le e^{\alpha T}\|Q^0\|.$$

Stability of upwind

The upwind method is stable in the 1-norm for $0 \leq \nu \leq 1$, where $\nu = uk/h$.

$$\begin{aligned} Q_i^{n+1} &= Q_i^n - \nu(Q_i^n - Q_{i-1}^n) \\ &= (1-\nu)Q_i^n + \nu Q_{i-1}^n \end{aligned}$$

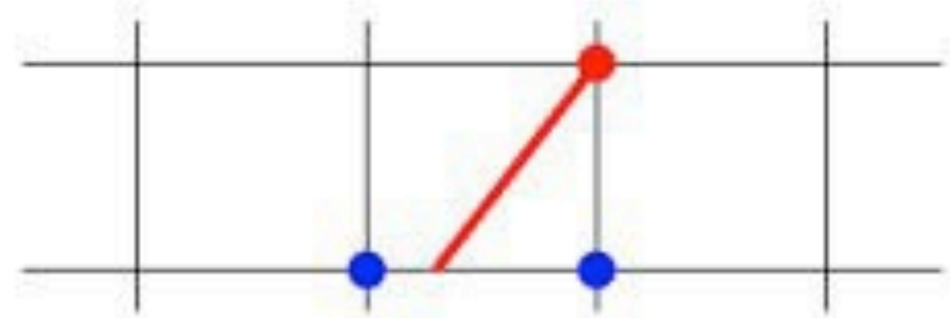
Note: convex combination if $0 \leq \nu \leq 1$

$$\begin{aligned} \|Q^{n+1}\|_1 &= h \sum_i |Q_i^{n+1}| \\ &= h \sum |(1-\nu)Q_i^n + \nu Q_{i-1}^n| \\ &\leq (1-\nu)h \sum |Q_i^n| + \nu h \sum |Q_i^n| \qquad \text{if } 0 \leq \nu \leq 1 \\ &= h \sum |Q_i^n| = \|Q^n\|_1 \end{aligned}$$

Upwind as interpolation

$q(x_i, t_{n+1}) = q(x_i - uk, t_n)$

Trace back along characteristic,
interpolate between grid values



Linear interpolation $\implies$ $q(x_i, t_{n+1}) \approx \nu Q_{i-1}^n + (1-\nu) Q_i^n$

Note: Upwind is **exact** if $uk/h = 1$,

$$Q_i^{n+1} = Q_{i-1}^n.$$

The CFL Condition

Domain of dependence: The solution $q(X,T)$ depends on the data $q(x,0)$ over some set of x values, $x \in \mathcal{D}(X,T)$.

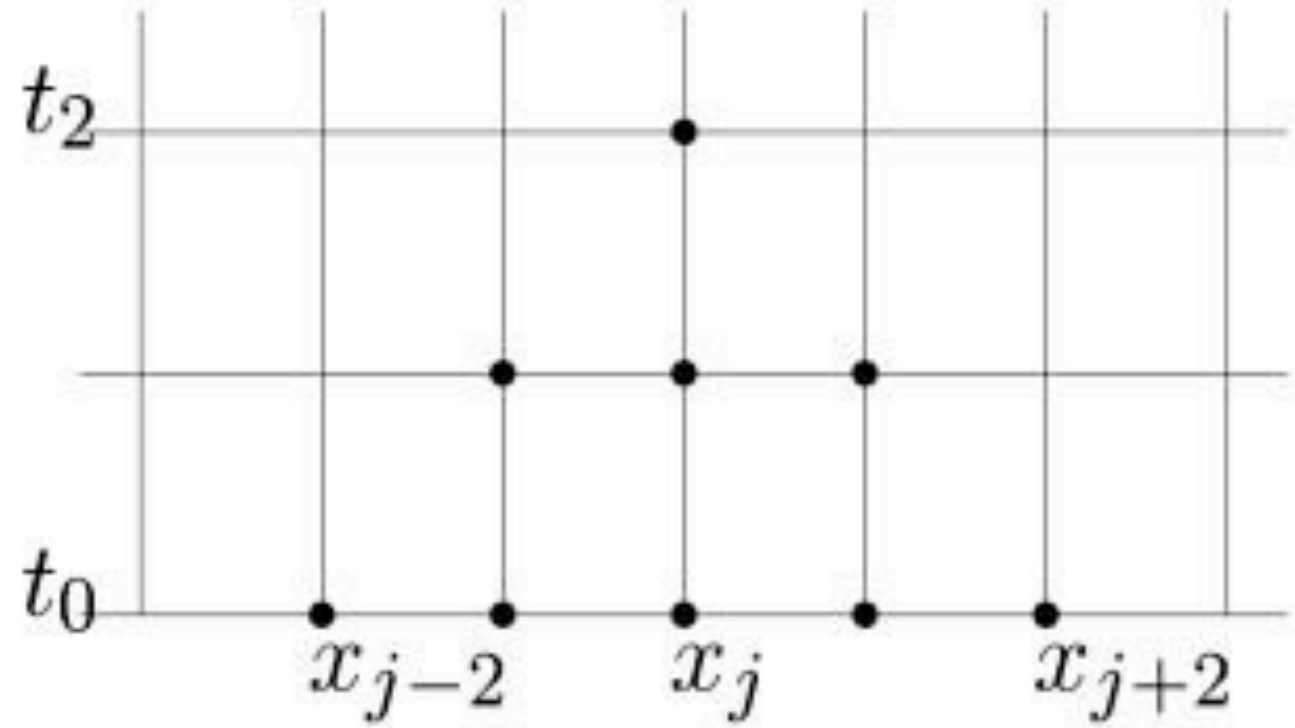
Advection: $q(X,T) = q(X - uT, 0)$ and so $\mathcal{D}(X,T) = \{X - uT\}$.

The CFL Condition: A numerical method can be convergent only if its numerical domain of dependence contains the true domain of dependence of the PDE, at least in the limit as k and h go to zero.

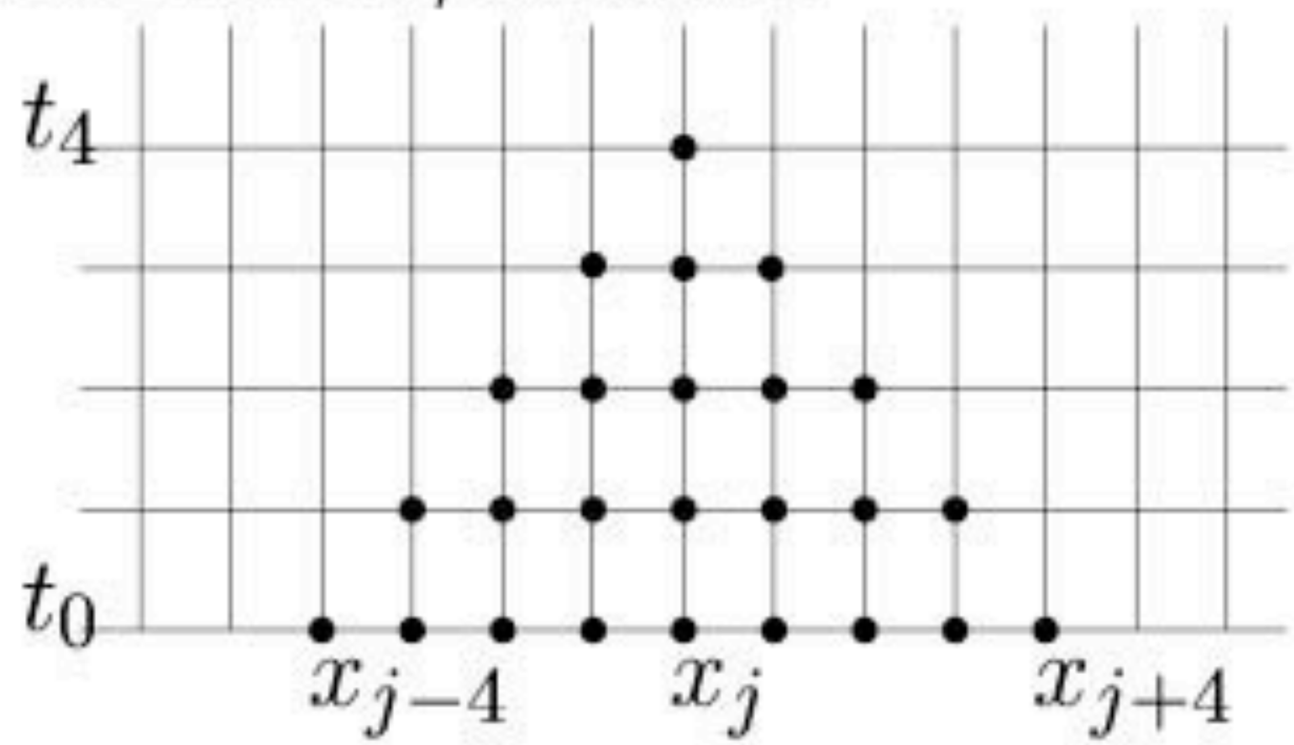
Note: Necessary but **not sufficient** for stability!

Numerical domain of dependence

With a 3-point explicit method:



On a finer grid with k/h fixed:



The CFL Condition

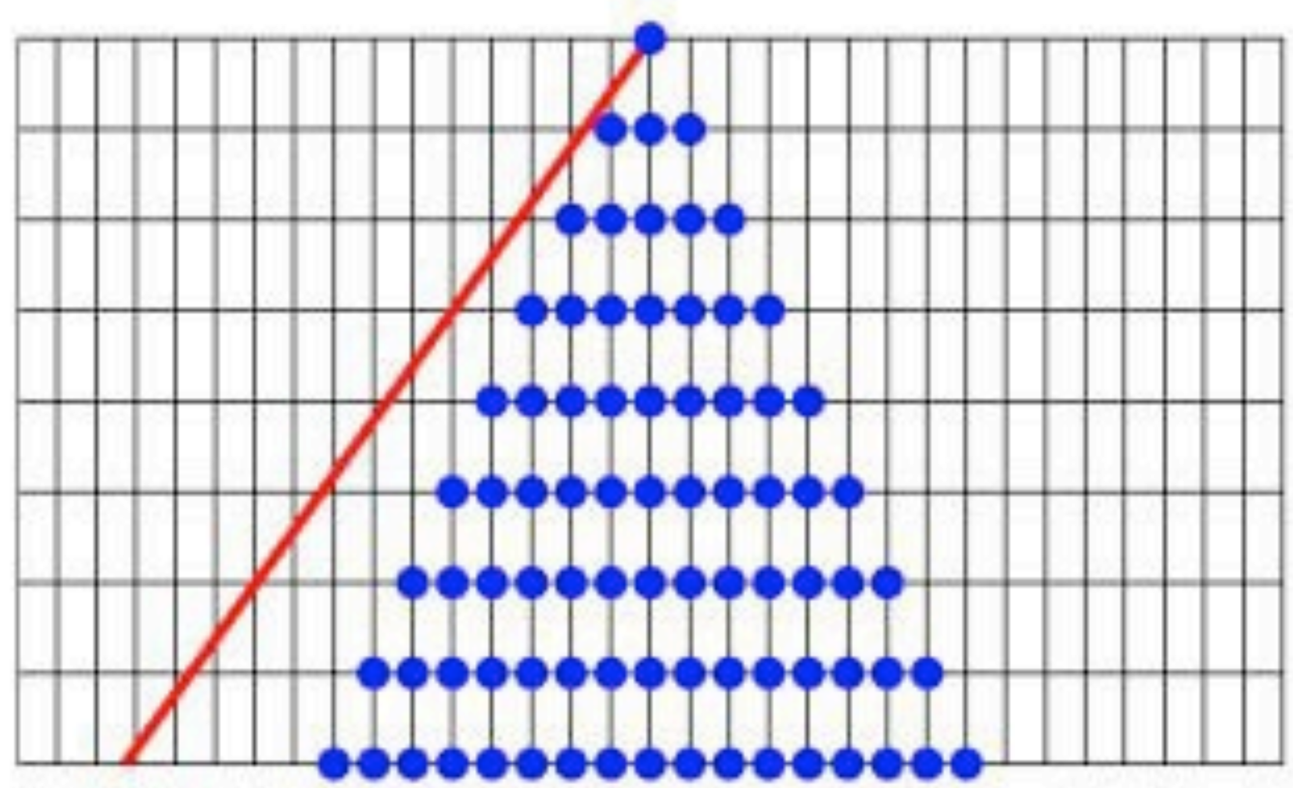
For the method to be stable, the numerical domain of dependence must include the true domain of dependence.

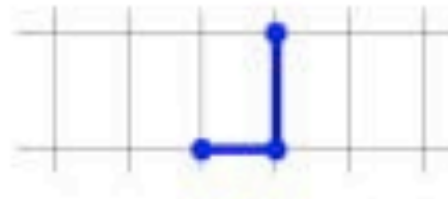
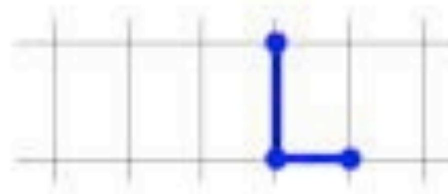
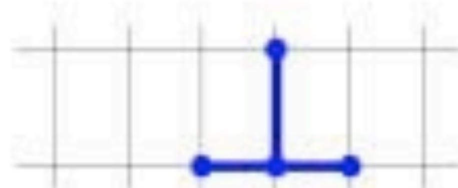
For advection, the solution is constant along characteristics,

$$q(x,t) = q(x - ut, 0)$$

For a 3-point method, CFL condition requires $\left|\frac{uk}{h}\right| \leq 1$.

If this is violated:



	Stencil	**CFL Condition**
		$0 \leq \frac{uk}{h} \leq 1$
		$-1 \leq \frac{uk}{h} \leq 0$
Lax-Wendroff		$-1 \leq \frac{uk}{h} \leq 1$
		$0 \leq \frac{uk}{h} \leq 2$

Hyperbolic systems

$$q_t + Aq_x = 0$$

A is $m \times m$ with eigenvalues λ^p and eigenvectors r^p, for $p = 1,\ 2,\ ,\ldots,\ m$.

Let R be matrix of right eigenvectors and $v = R^{-1}q$.

$$R^{-1}q_t + R^{-1}AR\,R^{-1}q_x = 0$$

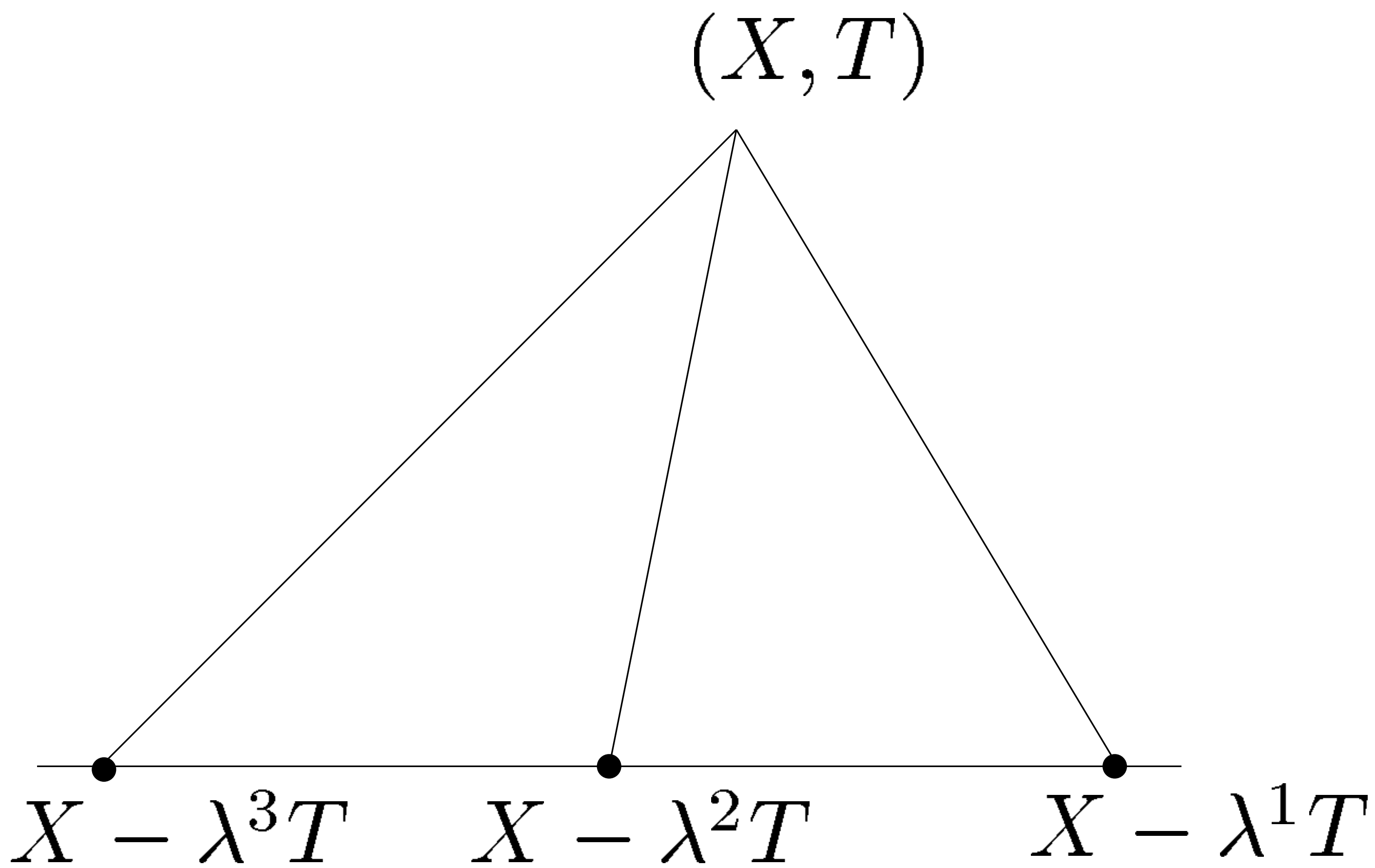
Since $R^{-1}AR = \Lambda$, this diagonalizes the system:

$$v_t + \Lambda v_x = 0.$$

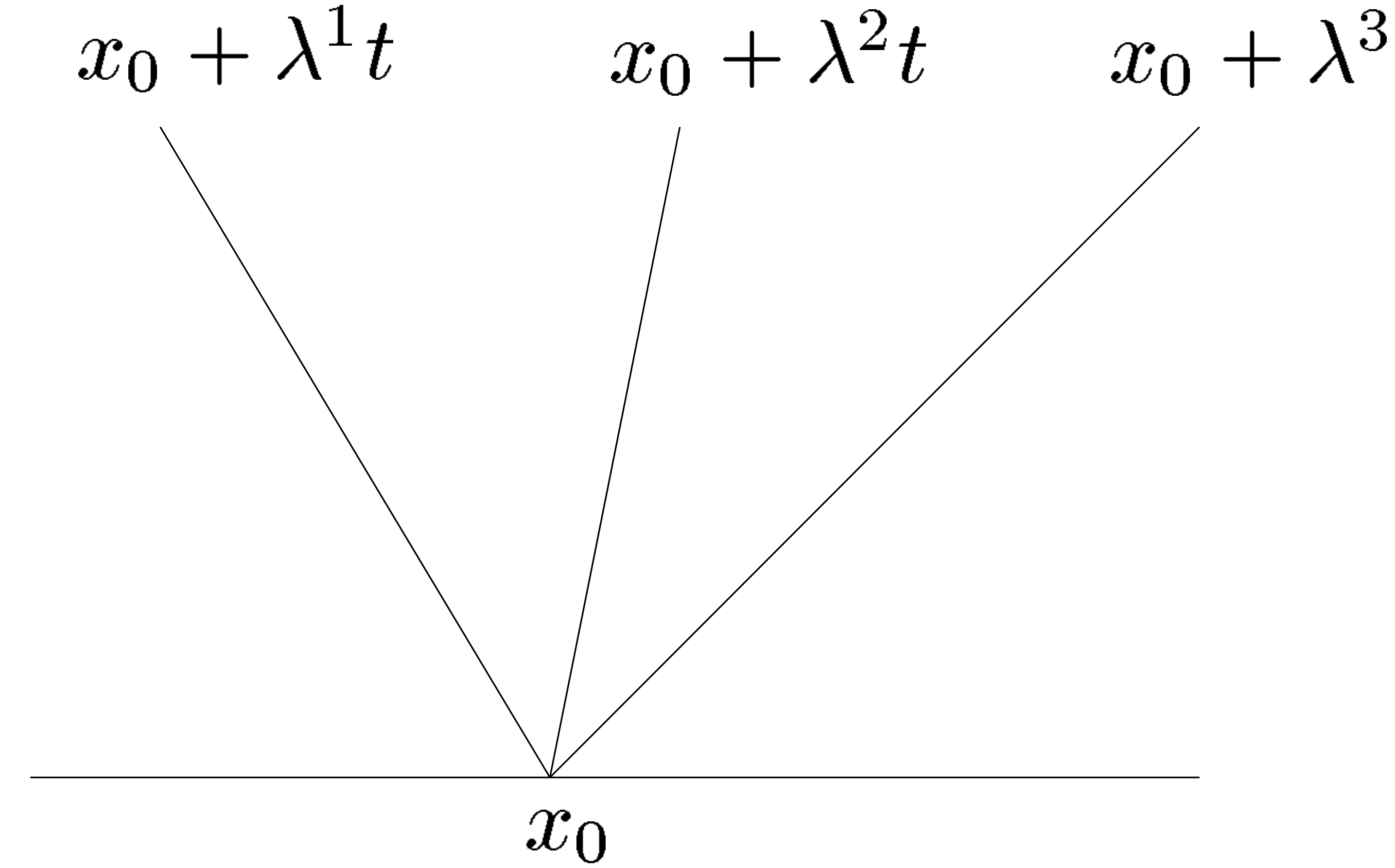
This is a system of m decoupled advection equations

$$v_t^p + \lambda^p v_x^p = 0.$$

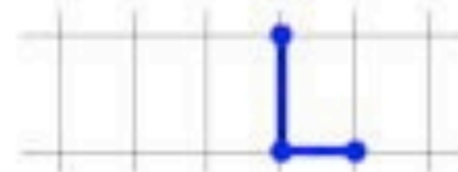
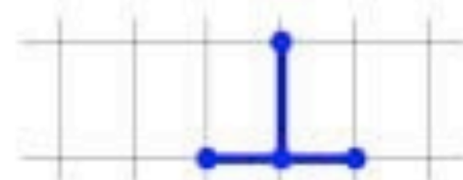
3 equations with $\lambda_1 < 0 < \lambda_2 < \lambda_3$



domain of dependence



range of infuence

Stencil	CFL Condition
	$0 \leq \dfrac{\lambda^p k}{h} \leq 1, \quad \forall p$
	$-1 \leq \dfrac{\lambda^p k}{h} \leq 0, \quad \forall p$
	$-1 \leq \dfrac{\lambda^p k}{h} \leq 1, \quad \forall p$
	$0 \leq \dfrac{\lambda^p k}{h} \leq 2, \quad \forall p$

Upwind for a linear system

The one-sided method

$$Q_i^{n+1} = Q_i^n - \frac{k}{h} A(Q_i^n - Q_{i-1}^n)$$

is stable only if $0 \le k\lambda^p/h \le 1$ for all p.

Upwind method based on sign of each λ^p:

Let $\lambda^+ = \max(\lambda, 0)$, $\lambda^- = \min(\lambda, 0)$,

$$\Lambda^+ = \text{diag}((\lambda^p)^+), \quad \Lambda^- = \text{diag}((\lambda^p)^-),$$

$$A^+ = R\Lambda^+ R^{-1}, \quad A^- = R\Lambda^- R^{-1}$$

Then

$$Q_i^{n+1} = Q_i^n - \frac{k}{h} A^+(Q_i^n - Q_{i-1}^n) - \frac{k}{h} A^-(Q_{i+1}^n - Q_i^n).$$

Symmetric methods

Centered in space, forward in time:

$$\begin{aligned} Q_i^{n+1} &= Q_i^n - \frac{k}{h}\left(\frac{1}{2}A\right)(Q_i^n - Q_{i-1}^n) - \frac{k}{h}\left(\frac{1}{2}A\right)(Q_{i+1}^n - Q_i^n) \\ &= Q_i^n - \frac{k}{2h}A(Q_{i+1}^n - Q_{i-1}^n) \end{aligned}$$

Centered approximation to q_x, but **unstable** for any fixed k/h.

Lax-Friedrichs:

$$Q_i^{n+1} = \frac{1}{2}(Q_{i-1}^n + Q_{i+1}^n) - \frac{k}{2h}A(Q_{i+1}^n - Q_{i-1}^n)$$

This is stable if $\left|\frac{\lambda^p k}{h}\right| \leq 1$ for all p.

Numerical dissipation

Lax-Friedrichs:

$$Q_i^{n+1} = \frac{1}{2}(Q_{i-1}^n + Q_{i+1}^n) - \frac{k}{2h}A(Q_{i+1}^n - Q_{i-1}^n)$$

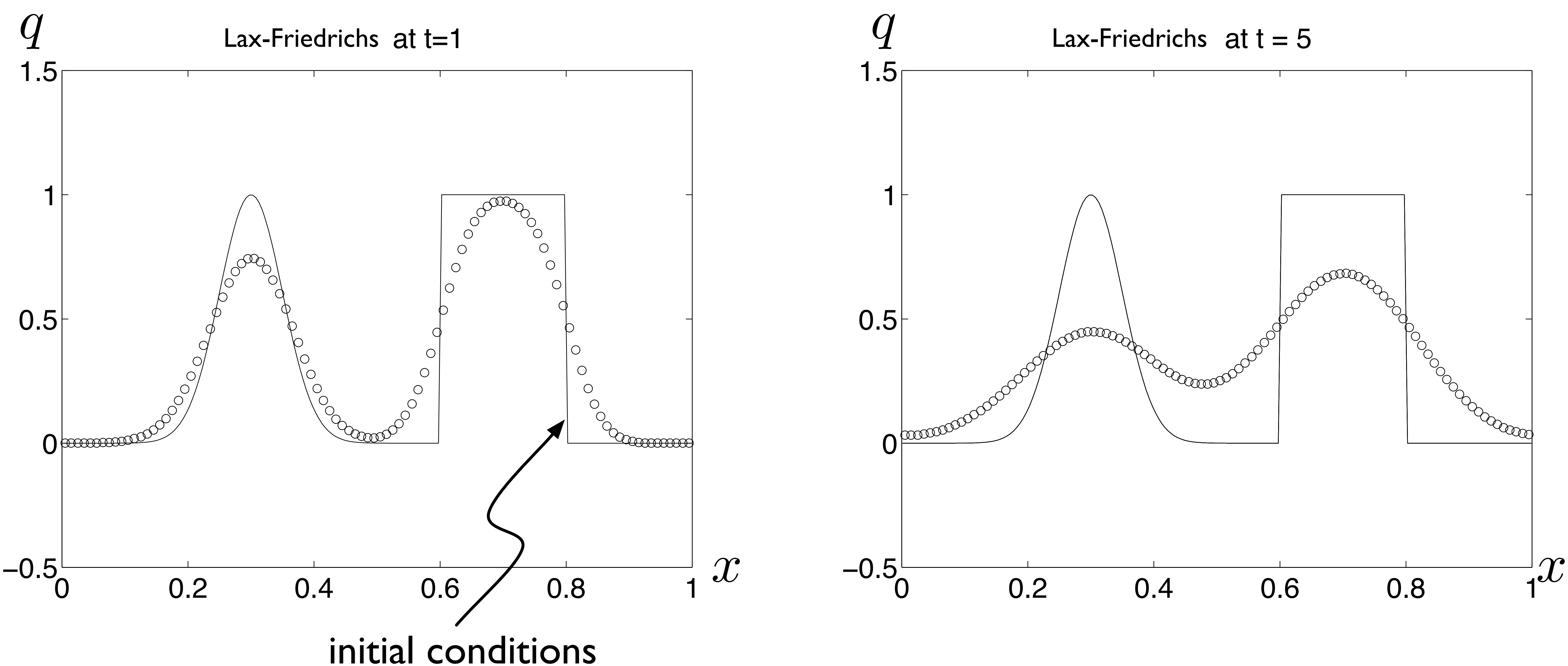
This can be rewritten as

$$Q_i^{n+1} = Q_i^n - \frac{k}{2h}A(Q_{i+1}^n - Q_{i-1}^n) + \frac{1}{2}(Q_{i-1}^n - 2Q_i^n + Q_{i+1}^n)$$

Looks like the unstable method with the addition of an approximation to $\frac{1}{2}h^2 q_{xx}$.

Approximates $q_t + Aq_x = \epsilon q_{xx}$. (modified equation)

$q_t + q_x = 0$ with periodic boundary conditions



Lax-Wendroff

Second-order accuracy?

Taylor series:

$$q(x, t+k) = q(x,t) + kq_t(x,t) + \frac{1}{2}k^2 q_{tt}(x,t) + \cdots$$

From $q_t = -Aq_x$ we find $q_{tt} = A^2 q_{xx}$.

$$q(x, t+k) = q(x,t) - kAq_x(x,t) + \frac{1}{2}k^2 A^2 q_{xx}(x,t) + \cdots$$

Replace q_x and q_{xx} by centered differences:

$$Q_i^{n+1} = Q_i^n - \frac{k}{2h}A(Q_{i+1}^n - Q_{i-1}^n) + \frac{1}{2}\frac{k^2}{h^2}A^2(Q_{i-1}^n - 2Q_i^n + Q_{i+1}^n)$$

Modified equations

The upwind method

$$Q_i^{n+1} = Q_i^n - \frac{k}{h} u(Q_i^n - Q_{i-1}^n).$$

gives a first-order accurate approximation to $q_t + uq_x = 0$.

But it gives a **second-order** approximation to

$$q_t + uq_x = \frac{uh}{2}\left(1 - \frac{uk}{h}\right) q_{xx}.$$

This is an advection-diffusion equation.

Indicates that the numerical solution will diffuse.

Note: coefficient of diffusive term is $O(h)$.

Modified Equation for Lax-Wendroff

The Lax-Wendroff method

$$Q_i^{n+1} = Q_i^n - \frac{k}{2h}A(Q_{i+1}^n - Q_{i-1}^n) + \frac{1}{2}\frac{k^2}{h^2}A^2(Q_{i-1}^n - 2Q_i^n + Q_{i+1}^n)$$

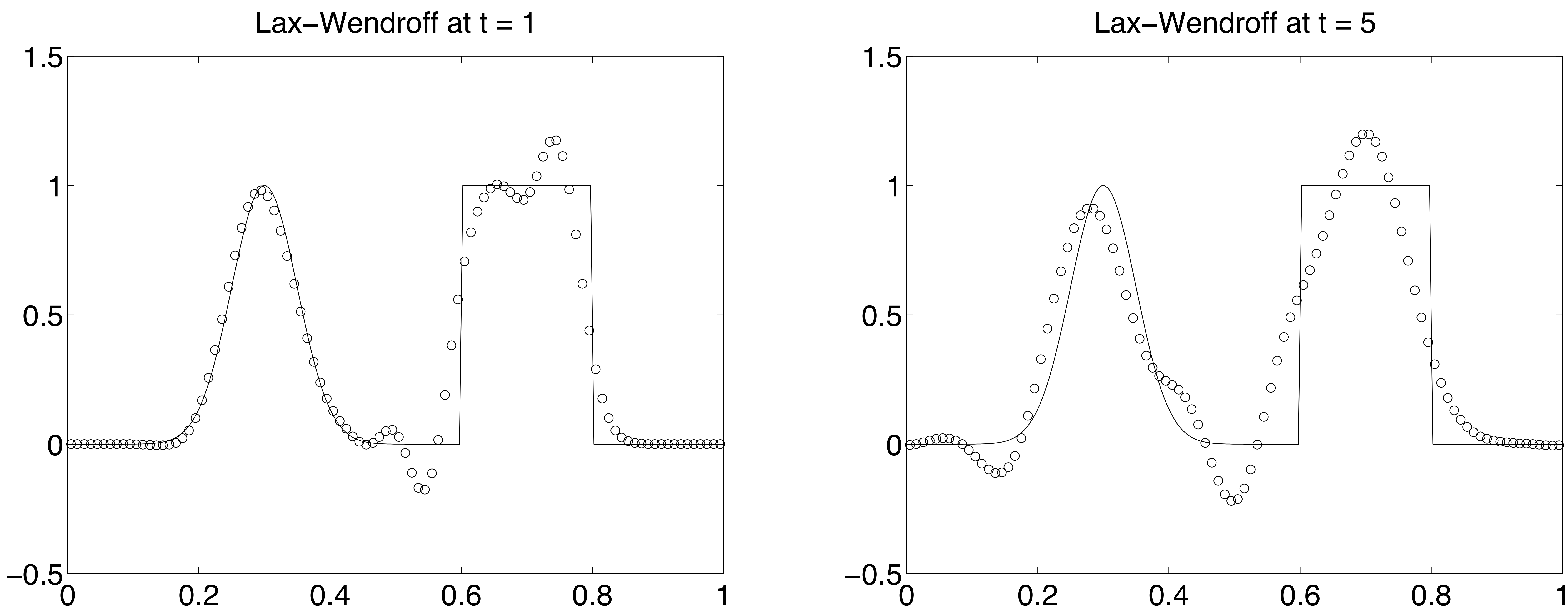
gives a second-order accurate approximation to $q_t + uq_x = 0$.

But it gives a **third-order** approximation to

$$q_t + uq_x = -\frac{uh^2}{6}\left(1 - \left(\frac{uk}{h}\right)^2\right)q_{xxx}.$$

This has a **dispersive** term with $O(h^2)$ coefficient.

Indicates that the numerical solution will become oscillatory.



Lax–Wendroff at t = 1
Lax–Wendroff at t = 5
1.5
1
0.5
0
−0.5
0
0.2
0.4
0.6
0.8
1

Dispersion relation

Consider a single Fourier mode:

$$q(x,0) = e^{i\xi x} \implies q(x,t) = e^{i(\xi x - \omega t)}$$

Determine $\omega(\xi)$ based on the PDE. This is the **dispersion relation**.

$$q_t = -i\omega q, \quad q_x = i\xi q, \quad q_{xx} = -\xi^2 q, \quad q_{xxx} = -i\xi^3 q, \ldots$$

$$q_t + uq_x = 0 \implies \omega(\xi) = u\xi, \qquad q(x,t) = e^{i\xi(x-ut)}$$

$$q_t + uq_x = \epsilon q_{xx} \implies \qquad q(x,t) = e^{-\epsilon\xi^2 t} e^{i\xi(x-ut)}$$

$$q_t + uq_x = \beta q_{xxx} \implies \qquad q(x,t) = e^{i\xi(x-(u+\beta\xi^2)t)}$$

Dispersive behavior

$$q_t + uq_x = \beta q_{xxx} \implies \qquad q(x,t) = e^{i\xi(x-(u+\beta\xi^2)t)}$$

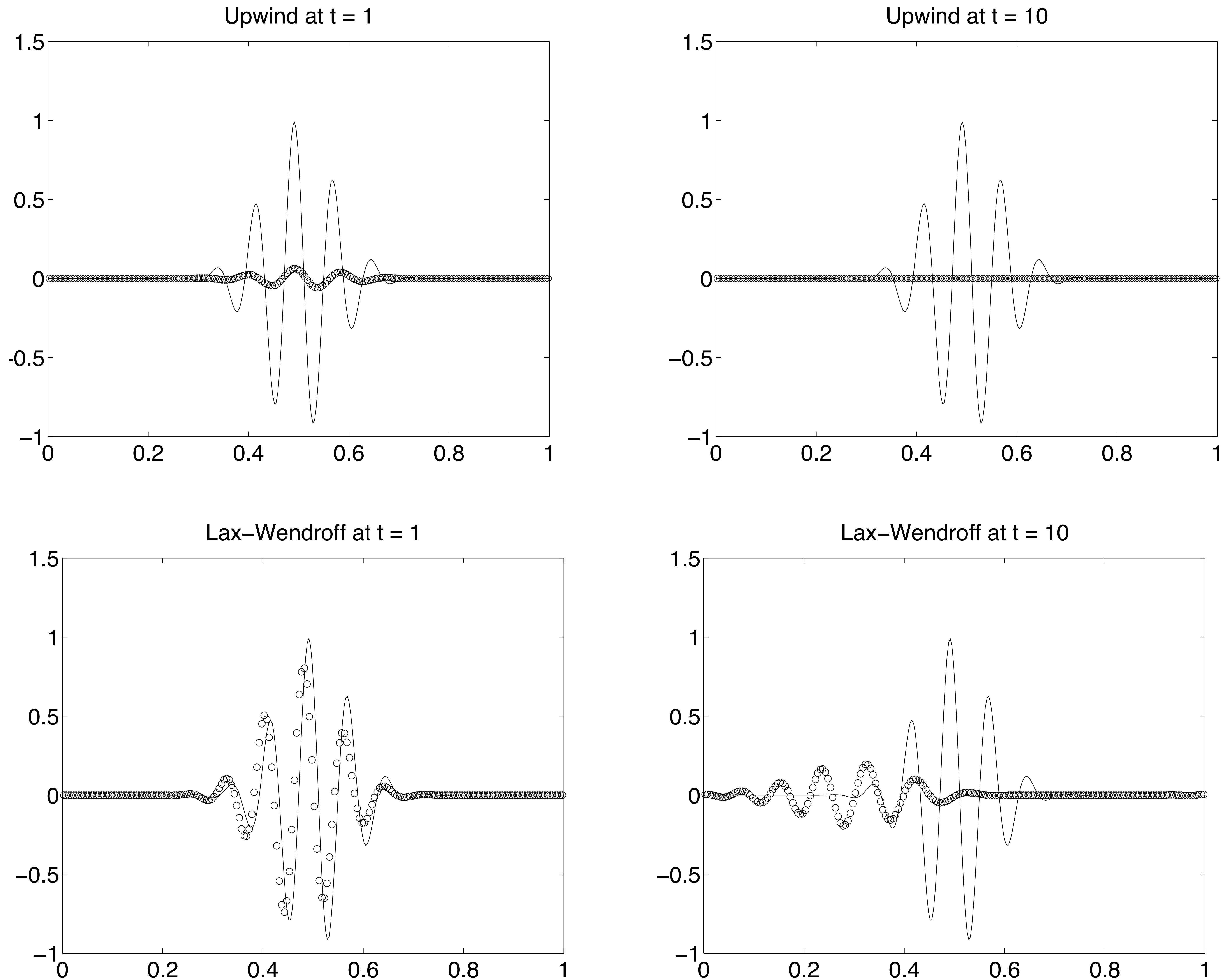
Dispersion relation: $\omega(\xi) = u\xi + \beta\xi^3$.

Wavenumber ξ propagates with **phase velocity**

$$c_p(\xi) = \frac{\omega(\xi)}{\xi} = u + \beta\xi^2.$$

Energy propagates with the **group velocity**

$$c_g(\xi) = \omega'(\xi) = u + 3\beta\xi^2.$$



Upwind at t = 1
Upwind at t = 10
Lax−Wendroff at t = 1
Lax−Wendroff at t = 10
1.5
1
0.5
0
−0.5
−1
0
0.2
0.4
0.6
0.8
1

Introduction to numerical methods to hyperbolic PDE's

Lecture 2: High resolution finite volume methods

download review from: http://www.mat.univie.ac.at/~obertsch/literatur/conservation_laws.pdf

Outline

- Finite volume methods
- Godunov's method
- High-resolution methods, TVD methods
- Slope limiters, flux limiters, wave limiters
- Nonlinear problems, convergence to weak solutions
- Conservation form, Lax-Wendroff theorem

Finite volume method

$$q_t + f(q)_x = 0$$

Integral form: $$\frac{\partial}{\partial t}\int_{x_{i-1/2}}^{x_{i+1/2}} q(x,t)\,dx = f(q(x_{i-1/2},t)) - f(q(x_{i+1/2},t))$$

Integrate from t_n to t_{n+1} $\Longrightarrow$

$$\int q(x,t_{n+1})\,dx = \int q(x,t_n)\,dx + \int_{t_n}^{t_{n+1}} f(q(x_{i-1/2},t)) - f(q(x_{i+1/2},t))\,dt$$

$$\frac{1}{h}\int q(x,t_{n+1})\,dx = \frac{1}{h}\int q(x,t_n)\,dx - \frac{k}{h}\left(\frac{1}{k}\int_{t_n}^{t_{n+1}} f(q(x_{i+1/2},t)) - f(q(x_{i-1/2},t))\,dt\right)$$

Numerical method: $$Q_i^{n+1} = Q_i^n - \frac{k}{h}(F_{i+1/2}^n - F_{i-1/2}^n) \qquad Q_i^n = \frac{1}{h}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} q(x,t_n)dx$$

Numerical flux: $F_{i-1/2}^n \approx \dfrac{1}{k}\displaystyle\int_{t_n}^{t_{n+1}} f(q(x_{i-1/2},t))\,dt.$

Finite volume method

$$Q_i^{n+1} = Q_i^n - \frac{k}{h}(F_{i+1/2}^n - F_{i-1/2}^n)$$

Advection equation: $f(q) = uq$

$$F_{i-1/2} \approx \frac{1}{k}\int_{t_n}^{t_{n+1}} uq(x_{i-1/2}, t)\, dt.$$

First order upwind:

$$F_{i-1/2} = u^+ Q_{i-1}^n + u^- Q_i^n$$

$$Q_i^{n+1} = Q_i^n - \frac{k}{h}(u^+(Q_i^n - Q_{i-1}^n) + u^-(Q_{i+1}^n - Q_i^n)).$$

where $u^+ = \max(u, 0)$, $u^- = \min(u, 0)$.

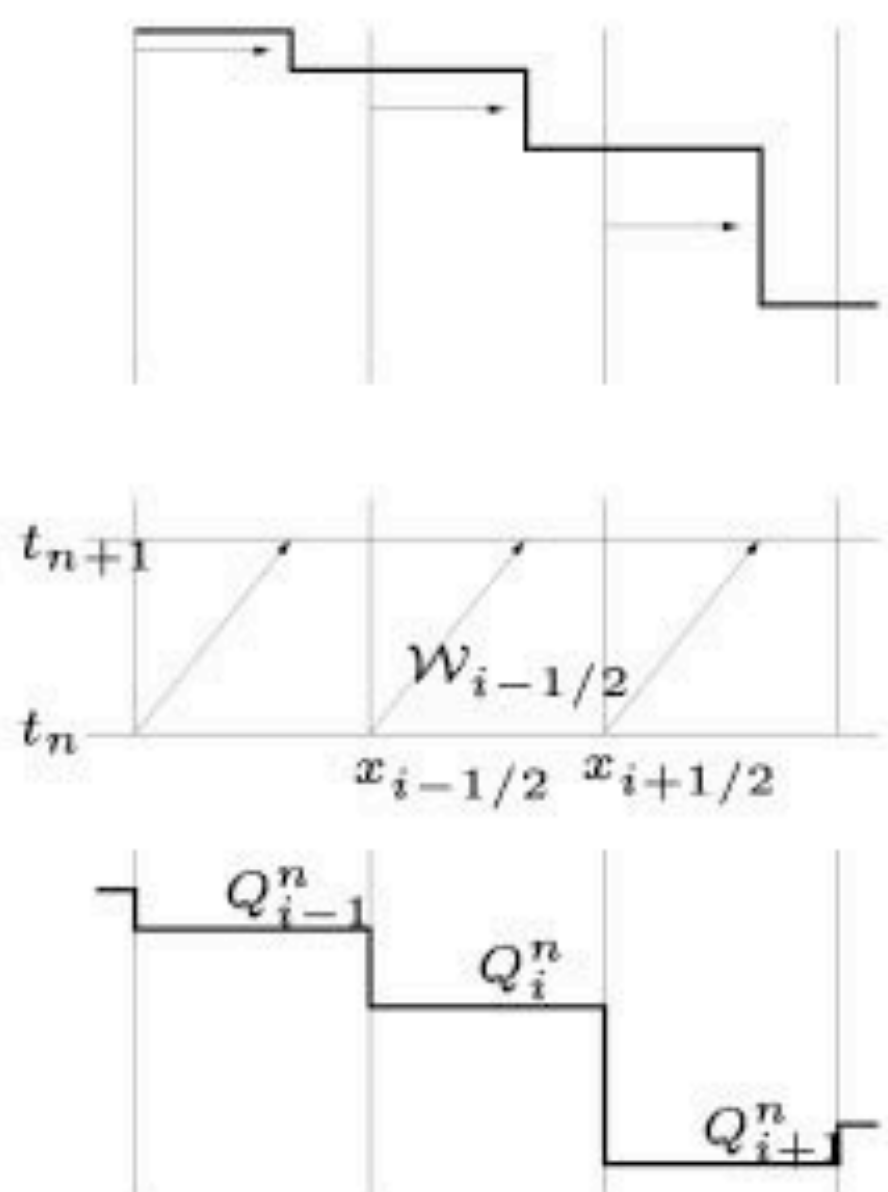
Godunov's method for advection

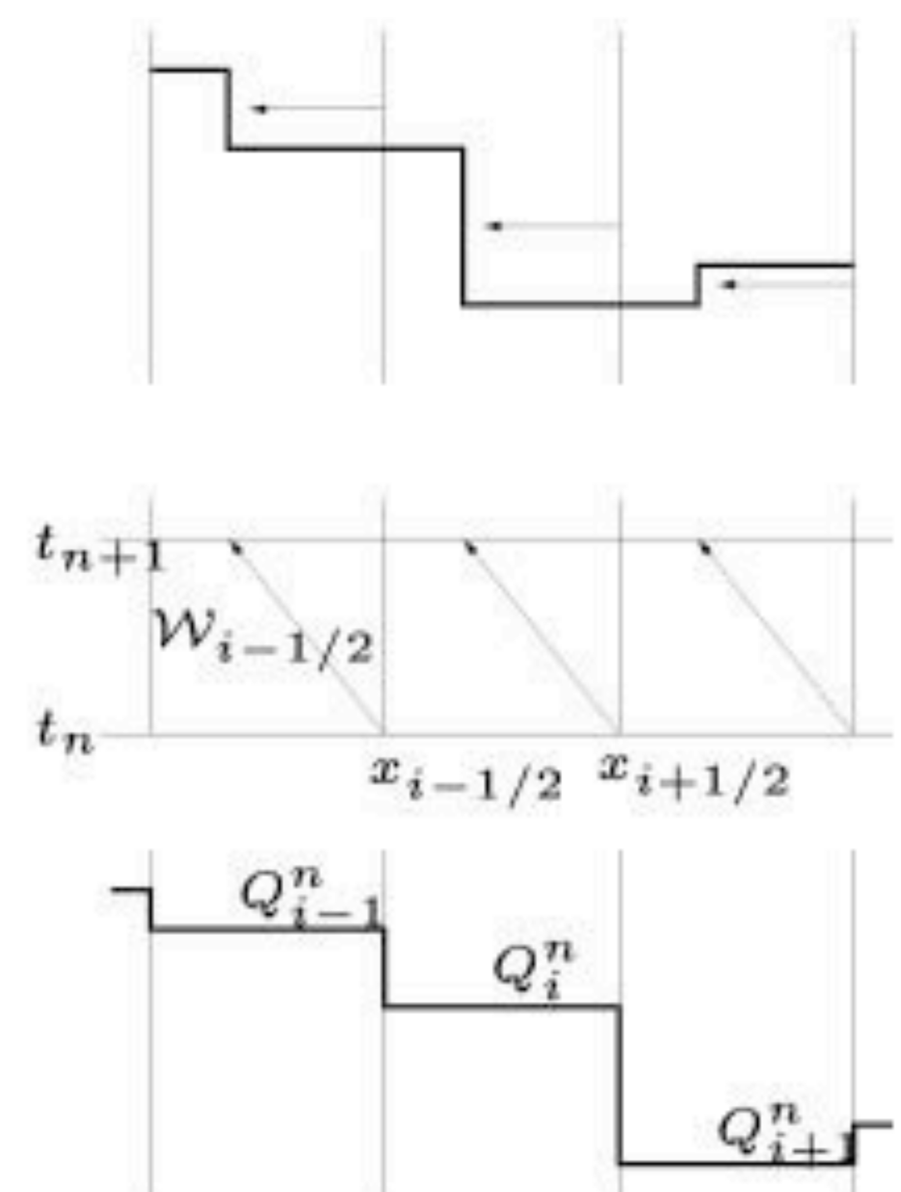
Q_i^n defines a piecewise constant function

$$\tilde{q}^n(x, t_n) = Q_i^n \text{ for } x_{i-1/2} < x < x_{i+1/2}$$

Discontinuities at cell interfaces $\Longrightarrow$ Riemann problems.

$u > 0$ $u < 0$



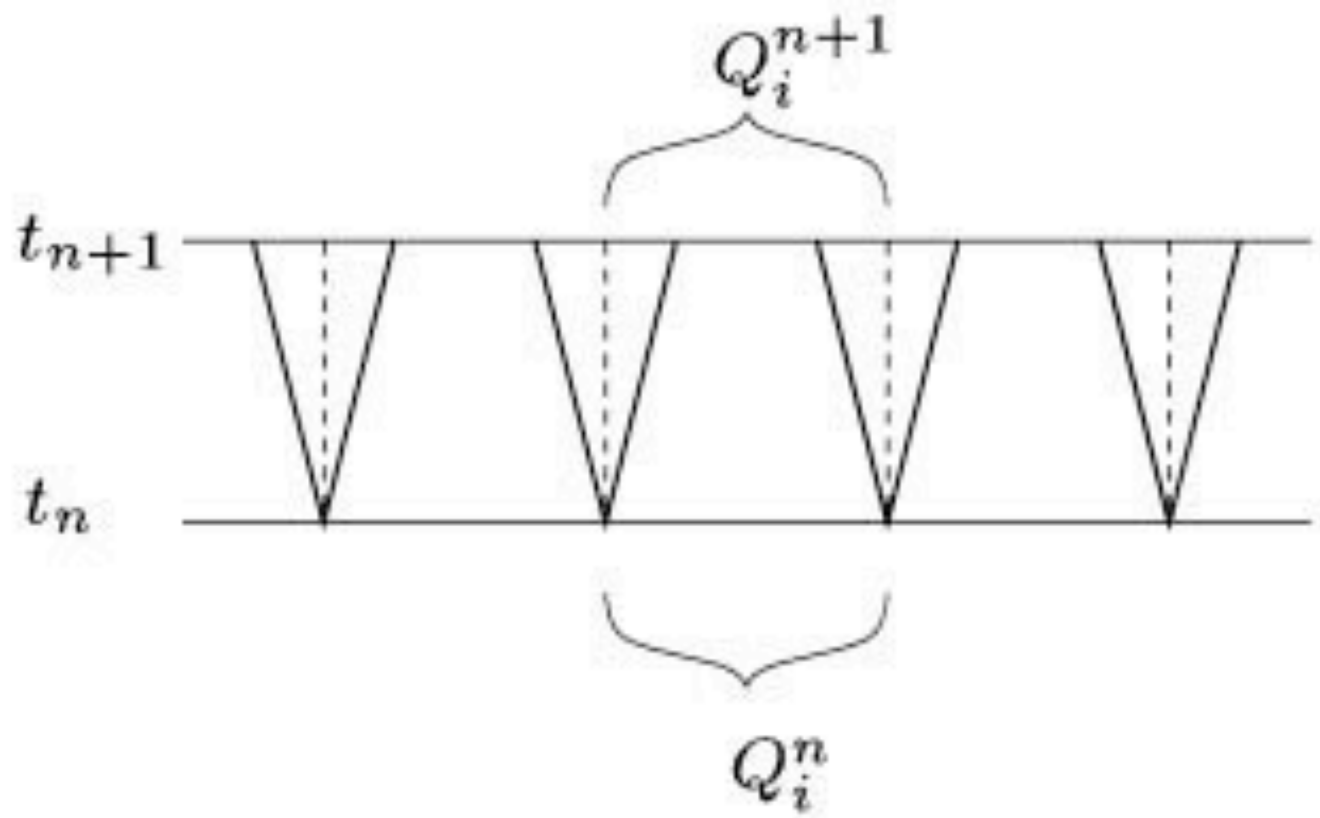


Godunov's method

Q_i^n defines a piecewise constant function

$$\tilde{q}^n(x, t_n) = Q_i^n \quad \text{for } x_{i-1/2} < x < x_{i+1/2}$$

Discontinuities at cell interfaces $\implies$ Riemann problems.



$$\tilde{q}^n(x_{i-1/2}, t) \equiv q^{\vee\!\!\!|}(Q_{i-1}, Q_i) \quad \text{for } t > t_n.$$

$$F_{i-1/2}^n = \frac{1}{k} \int_{t_n}^{t_{n+1}} f(q^{\vee\!\!\!|}(Q_{i-1}^n, Q_i^n))\, dt = f(q^{\vee\!\!\!|}(Q_{i-1}^n, Q_i^n)).$$

First order REA Algorithm

1. **Reconstruct** a piecewise constant function $\tilde{q}^n(x, t_n)$ defined for all x, from the cell averages Q_i^n.

$$\tilde{q}^n(x, t_n) = Q_i^n \quad \text{for all } x \in \mathcal{C}_i.$$

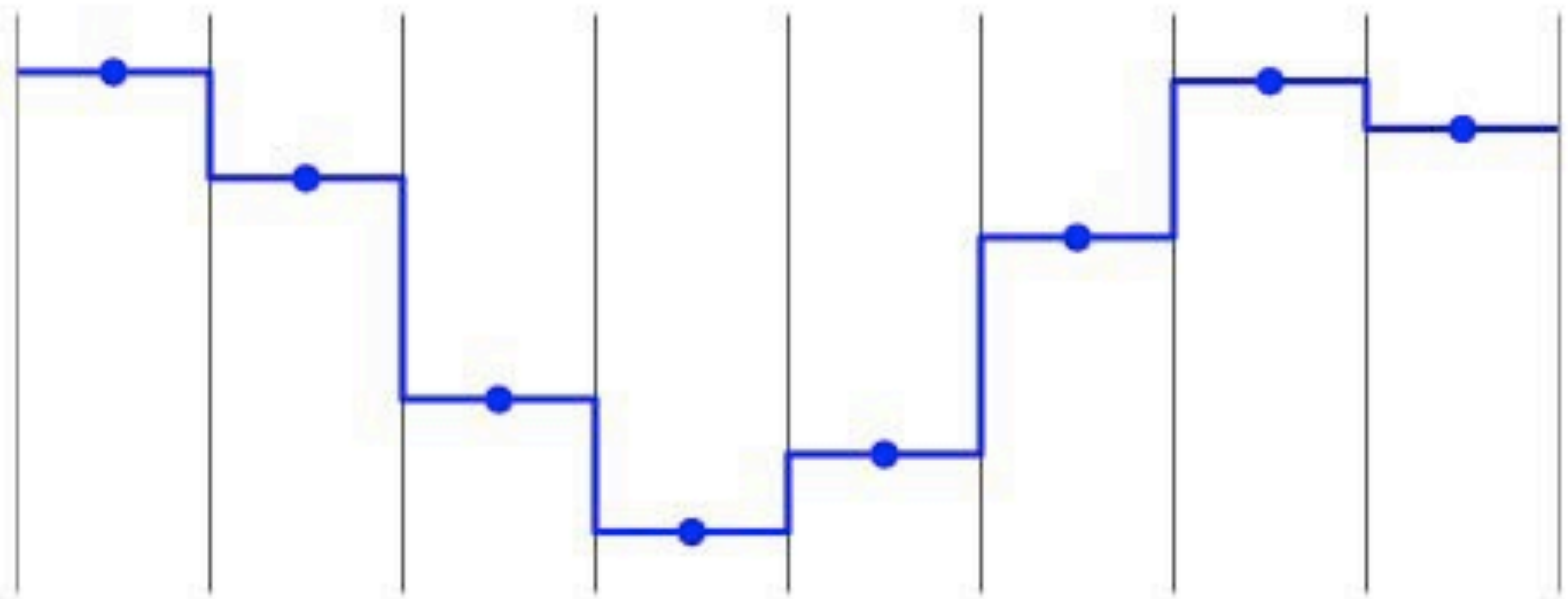
2. **Evolve** the hyperbolic equation exactly (or approximately) with this initial data to obtain $\tilde{q}^n(x, t_{n+1})$ a time k later.

3. **Average** this function over each grid cell to obtain new cell averages

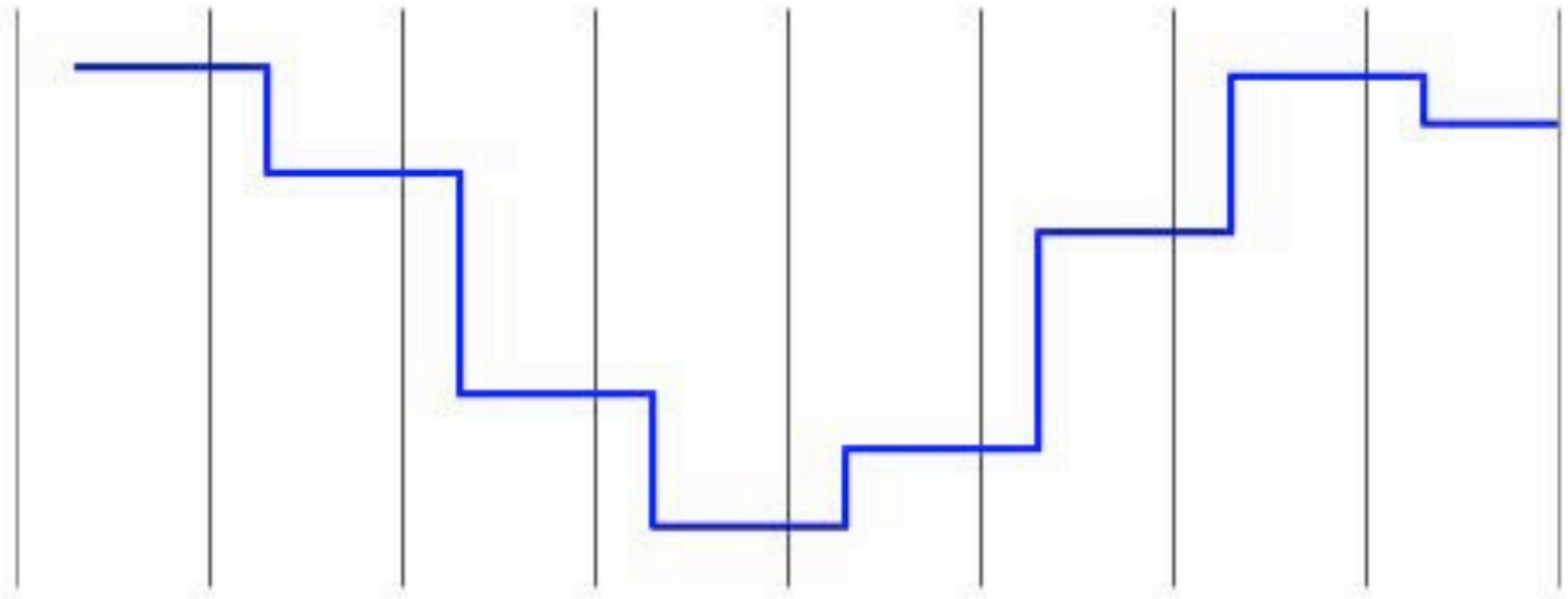
$$Q_i^{n+1} = \frac{1}{h} \int_{\mathcal{C}_i} \tilde{q}^n(x, t_{n+1})\, dx.$$

First order REA Algorithm

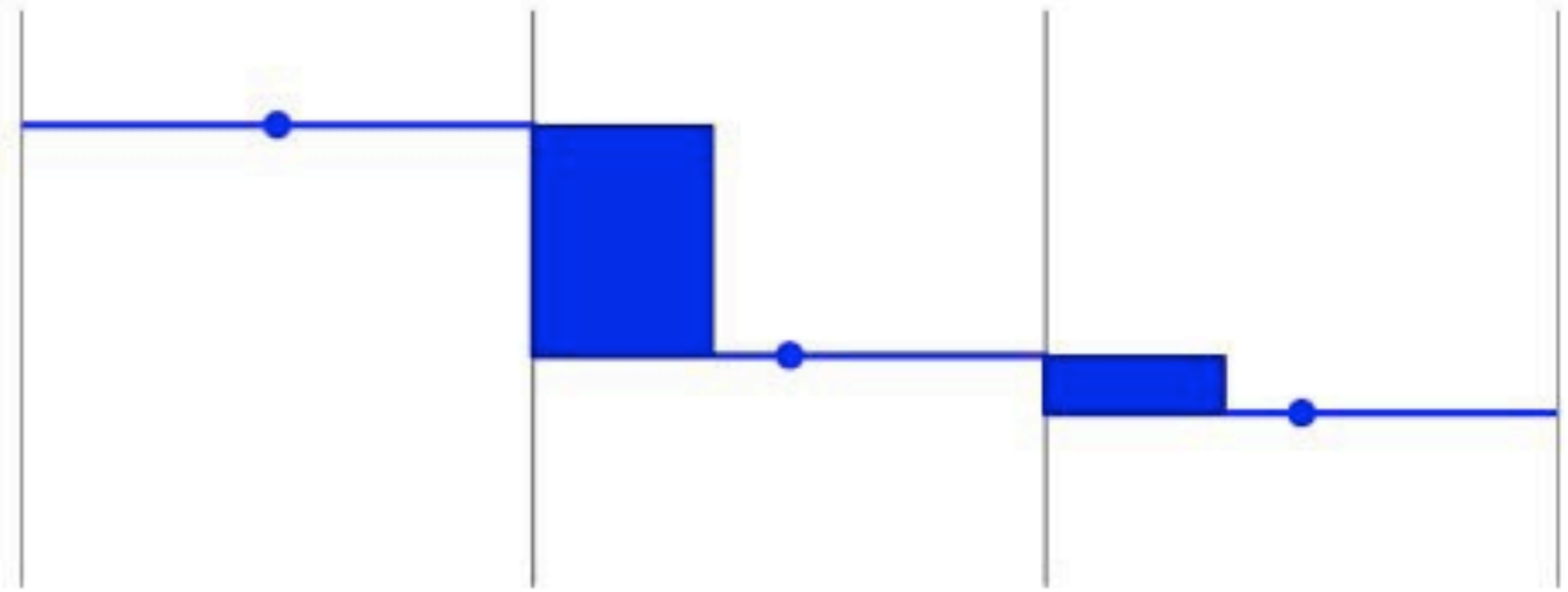
Cell averages and piecewise constant reconstruction:



After evolution:



Cell update



The cell average is modified by

$$\frac{ku \cdot (Q_{i-1}^n - Q_i^n)}{h}$$

So we obtain the upwind method

$$Q_i^{n+1} = Q_i^n - \frac{ku}{h}(Q_i^n - Q_{i-1}^n).$$

Second-order REA Algorithm

1. **Reconstruct** a piecewise **linear** function $\tilde{q}^n(x, t_n)$ defined for all x, from the cell averages Q_i^n.

$$\tilde{q}^n(x, t_n) = Q_i^n + \sigma_i^n (x - x_i) \quad \text{for all } x \in \mathcal{C}_i.$$

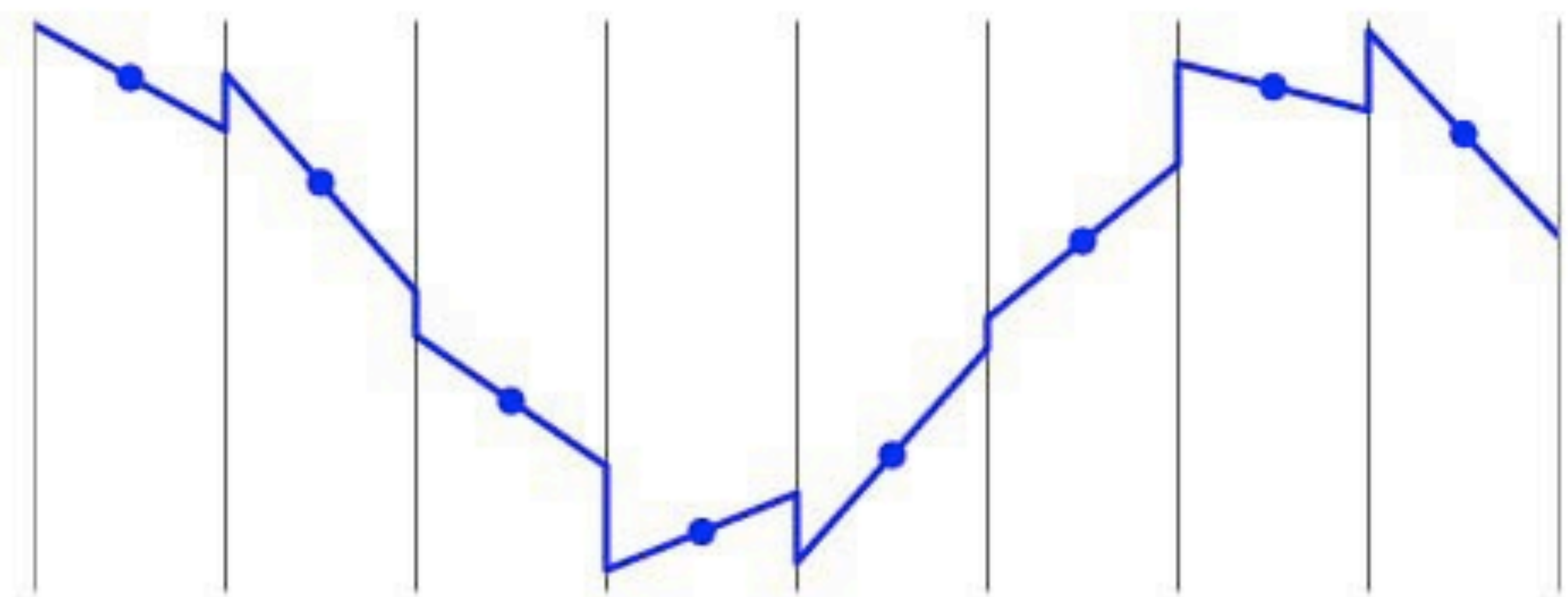
2. **Evolve** the hyperbolic equation exactly (or approximately) with this initial data to obtain $\tilde{q}^n(x, t_{n+1})$ a time k later.

3. **Average** this function over each grid cell to obtain new cell averages

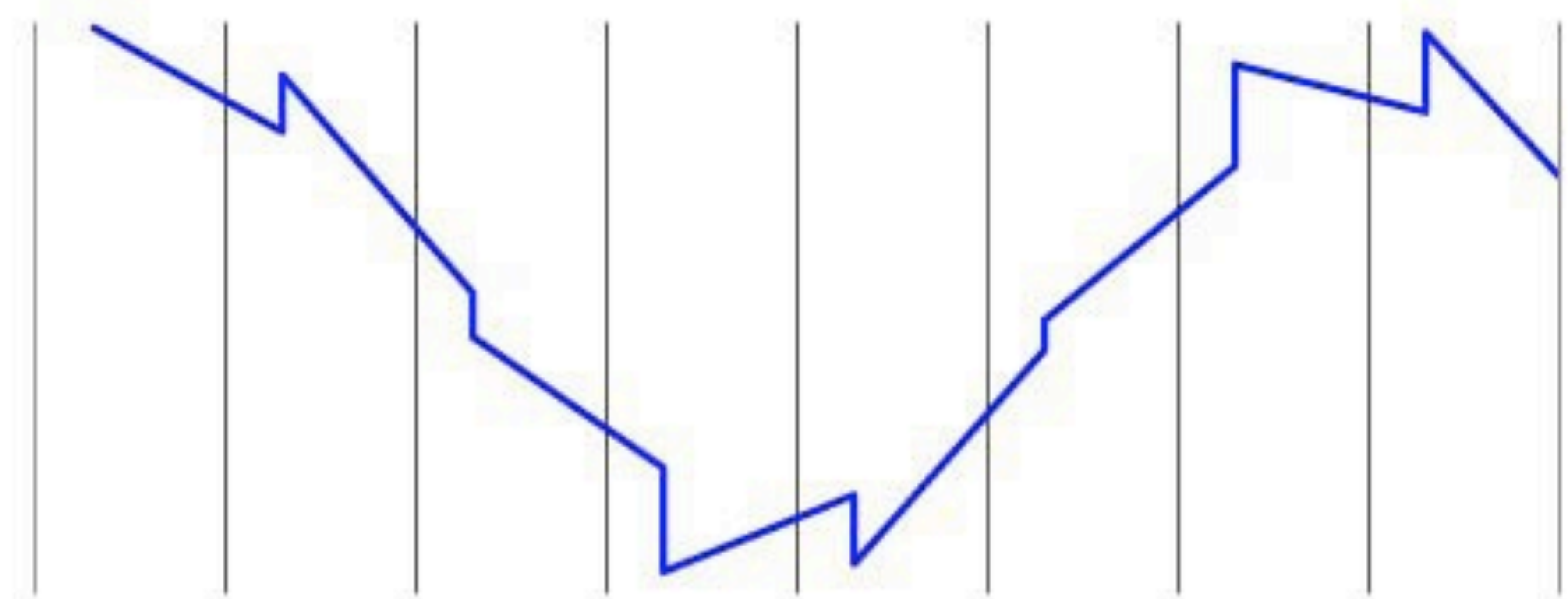
$$Q_i^{n+1} = \frac{1}{h} \int_{\mathcal{C}_i} \tilde{q}^n(x, t_{n+1})\, dx.$$

Second-order REA Algorithm

Cell averages and piecewise linear reconstruction:



After evolution:



Choice of slopes

$$\tilde{Q}^n(x, t_n) = Q_i^n + \sigma_i^n (x - x_i) \qquad \text{for } x_{i-1/2} \leq x < x_{i+1/2}.$$

Applying REA algorithm gives:

$$Q_i^{n+1} = Q_i^n - \frac{ku}{h}(Q_i^n - Q_{i-1}^n) - \frac{1}{2}\frac{ku}{h}(h - uk)(\sigma_i^n - \sigma_{i-1}^n)$$

Choice of slopes:

Centered slope: $\sigma_i^n = \dfrac{Q_{i+1}^n - Q_{i-1}^n}{2h}$ (Fromm)

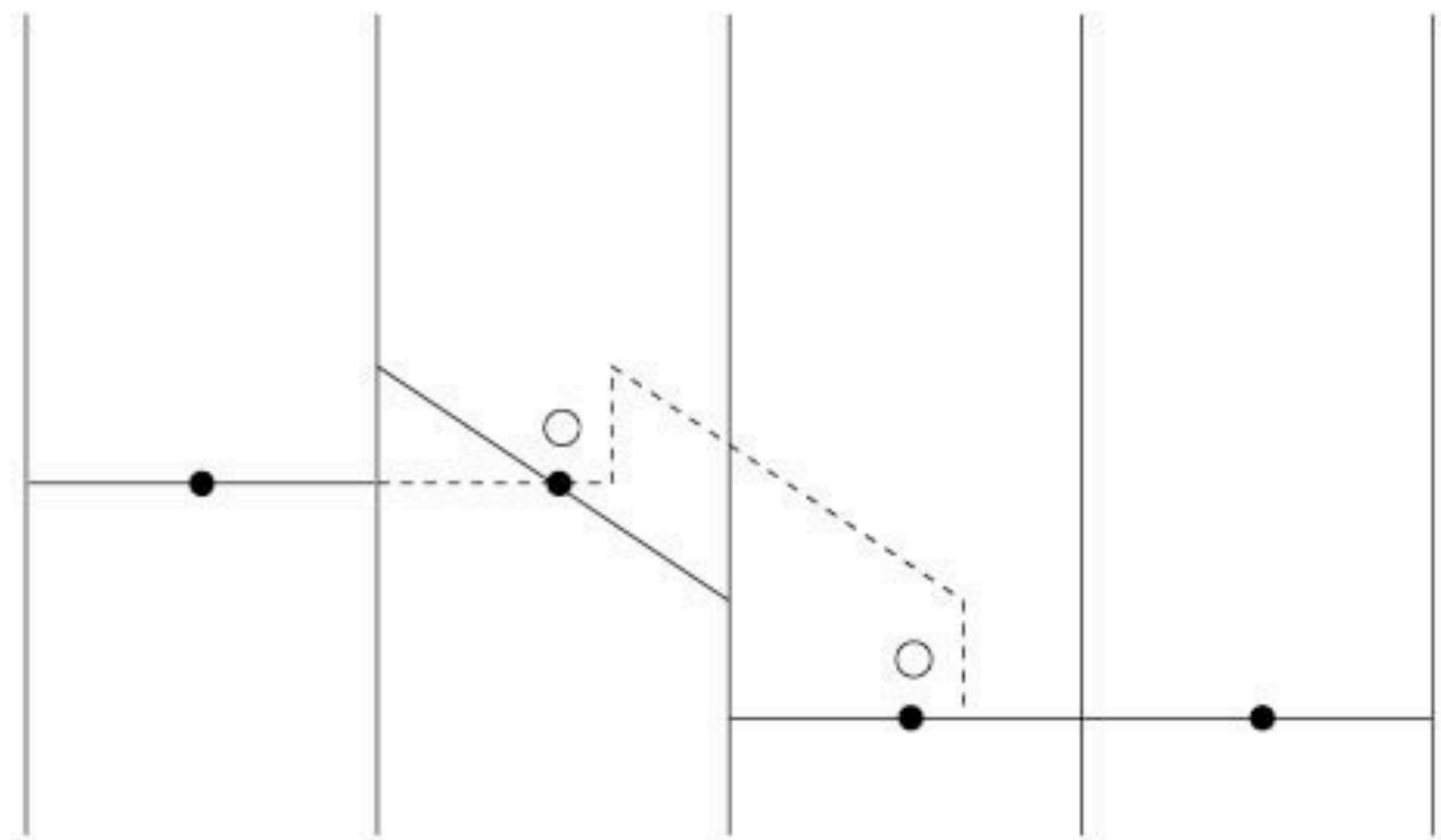
Upwind slope: $\sigma_i^n = \dfrac{Q_i^n - Q_{i-1}^n}{h}$ (Beam-Warming)

Downwind slope: $\sigma_i^n = \dfrac{Q_{i+1}^n - Q_i^n}{h}$ (Lax-Wendroff)

Oscillations

Any of these slope choices will give oscillations near discontinuities.

Ex: Lax-Wendroff:



High-resolution methods

Want to use slope where solution is smooth for “second-order” accuracy.

Where solution is not smooth, adding slope corrections gives oscillations.

Limit the slope based on the behavior of the solution.

$$\sigma_i^n = \left(\frac{Q_{i+1}^n - Q_i^n}{h}\right)\Phi_i^n.$$

$\Phi = 1 \implies$ Lax-Wendroff,

$\Phi = 0 \implies$ upwind.

Minmod slope

$$\text{minmod}(a,b) = \begin{cases} a & \text{if } |a| < |b| \text{ and } ab > 0 \\ b & \text{if } |b| < |a| \text{ and } ab > 0 \\ 0 & \text{if } ab \le 0 \end{cases}$$

Slope:

$$\begin{aligned} \sigma_i^n &= \text{minmod}((Q_i^n - Q_{i-1}^n)/h,\ \ (Q_{i+1}^n - Q_i^n)/h) \\ &= \left(\frac{Q_{i+1}^n - Q_i^n}{h}\right)\Phi(\theta_i^n) \end{aligned}$$

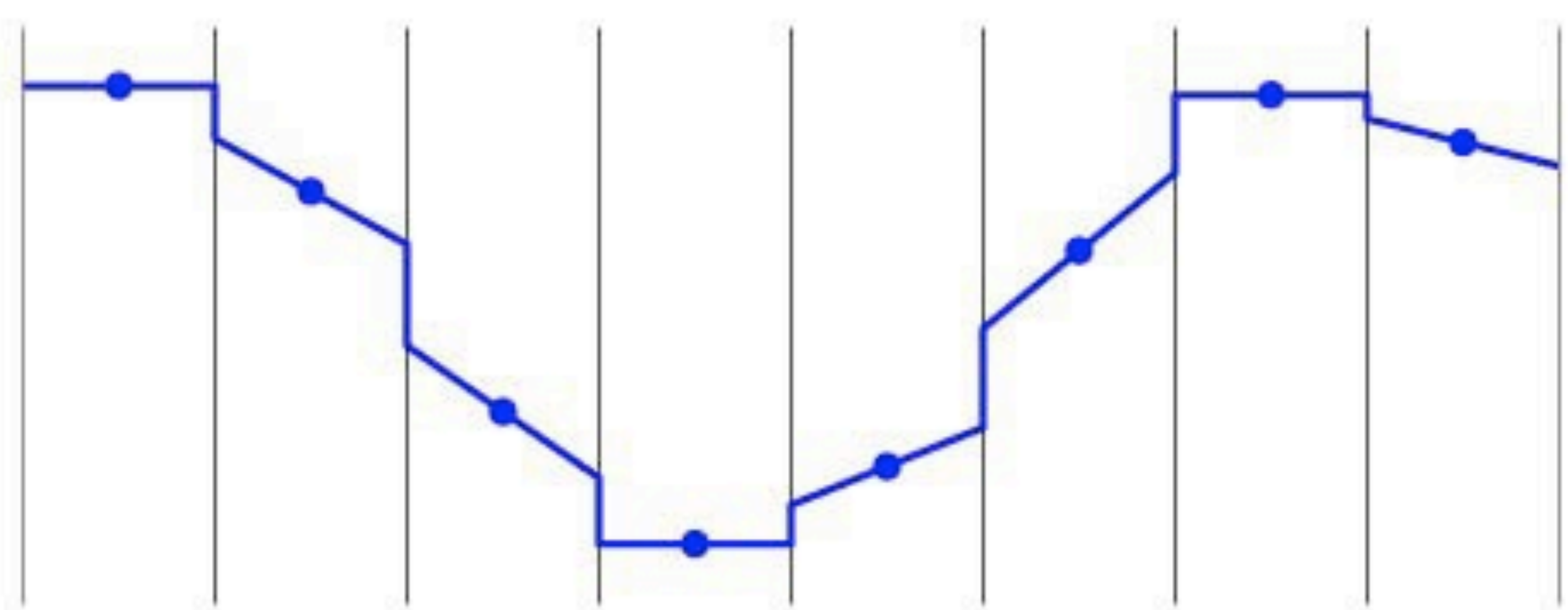
where

$$\begin{aligned} \theta_i^n &= \frac{Q_i^n - Q_{i-1}^n}{Q_{i+1}^n - Q_i^n} \\ \Phi(\theta) &= \text{minmod}(\theta, 1) \end{aligned}$$

Piecewise linear reconstruction

Lax-Wendroff reconstruction:

Minmod reconstruction:



TVD Methods

Total variation:

$$TV(Q) = \sum_i |Q_i - Q_{i-1}|$$

For a function, $TV(q) = \int |q_x(x)|\, dx$.

A method is **Total Variation Diminishing (TVD)** if

$$TV(Q^{n+1}) \leq TV(Q^n).$$

If Q^n is monotone, then so is Q^{n+1}.

No spurious oscillations generated.

Gives a form of stability useful for proving convergence, also for **nonlinear scalar** conservation laws.

TVD REA Algorithm

1. **Reconstruct** a piecewise **linear** function $\tilde{q}^n(x, t_n)$ defined for all x, from the cell averages Q_i^n.

$$\tilde{q}^n(x, t_n) = Q_i^n + \sigma_i^n(x - x_i) \quad \text{for all } x \in \mathcal{C}_i$$

with the property that $TV(\tilde{q}^n) \leq TV(Q^n)$.

2. **Evolve** the hyperbolic equation exactly (or approximately) with this initial data to obtain $\tilde{q}^n(x, t_{n+1})$ a time k later.

3. **Average** this function over each grid cell to obtain new cell averages

$$Q_i^{n+1} = \frac{1}{h} \int_{\mathcal{C}_i} \tilde{q}^n(x, t_{n+1})\, dx.$$

Note: Steps 2 and 3 are always TVD.

Some popular limiters

Linear methods:

$$\theta_j = \frac{U_j - U_{j-1}}{U_{j+1} - U_j}$$

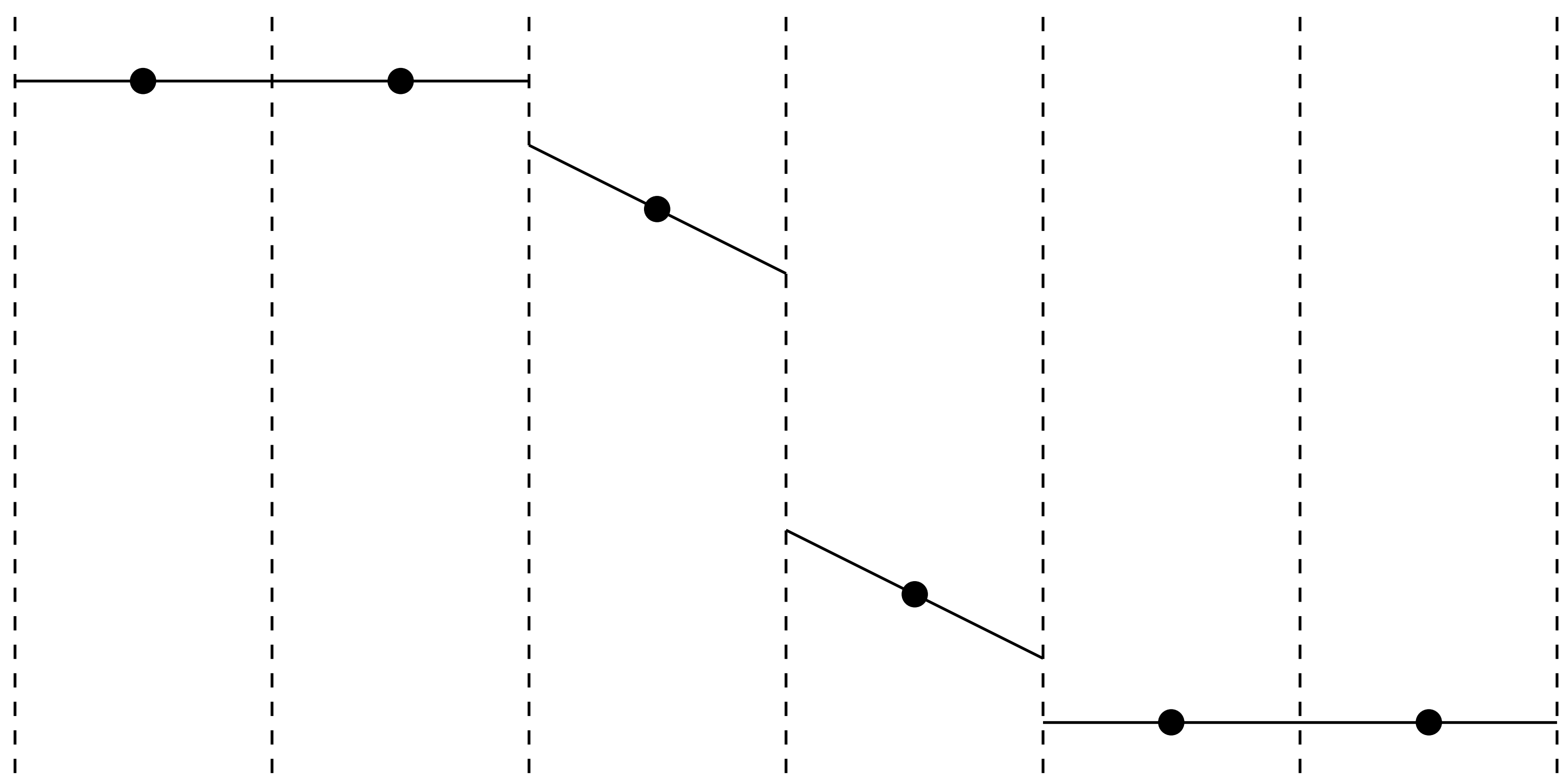
$$\begin{aligned}
\text{upwind}: &\quad \phi(\theta) = 0 \\
\text{Lax-Wendroff}: &\quad \phi(\theta) = 1 \\
\text{Beam-Warming}: &\quad \phi(\theta) = \theta \\
\text{Fromm}: &\quad \phi(\theta) = \frac{1}{2}(1+\theta)
\end{aligned}$$

High-resolution limiters:

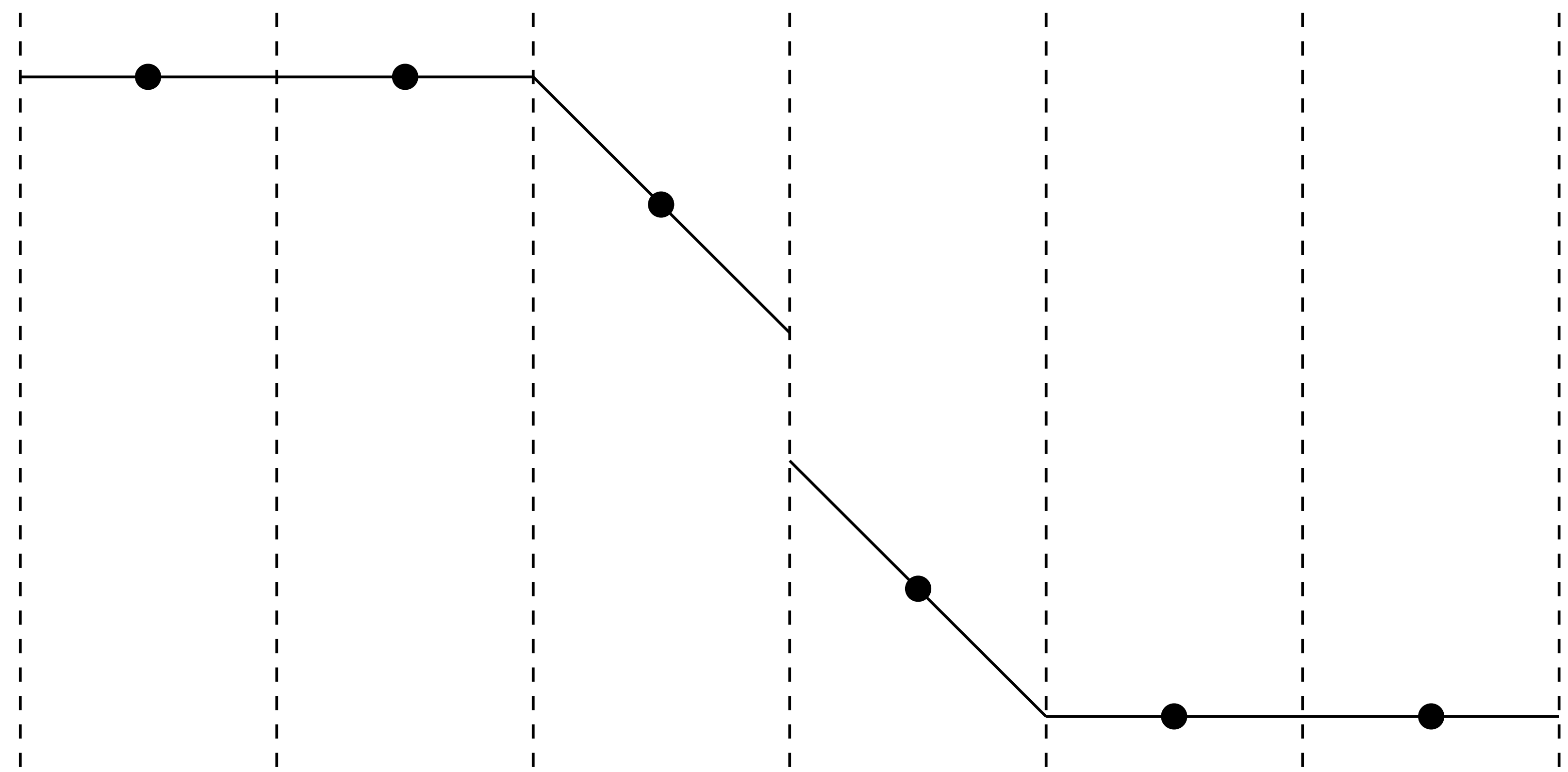
$$\begin{aligned}
\text{minmod}: &\quad \phi(\theta) = \text{minmod}(1, \theta) \\
\text{superbee}: &\quad \phi(\theta) = \max(0,\ \min(1, 2\theta),\ \min(2, \theta)) \\
\text{MC}: &\quad \phi(\theta) = \max(0,\ \min((1+\theta)/2,\ 2,\ 2\theta)) \\
\text{van Leer}: &\quad \phi(\theta) = \frac{\theta + |\theta|}{1 + |\theta|}
\end{aligned}$$

Piecewise linear reconstruction

Grid values Q^n and reconstructed $\tilde{q}^n(\cdot, t_n)$ using



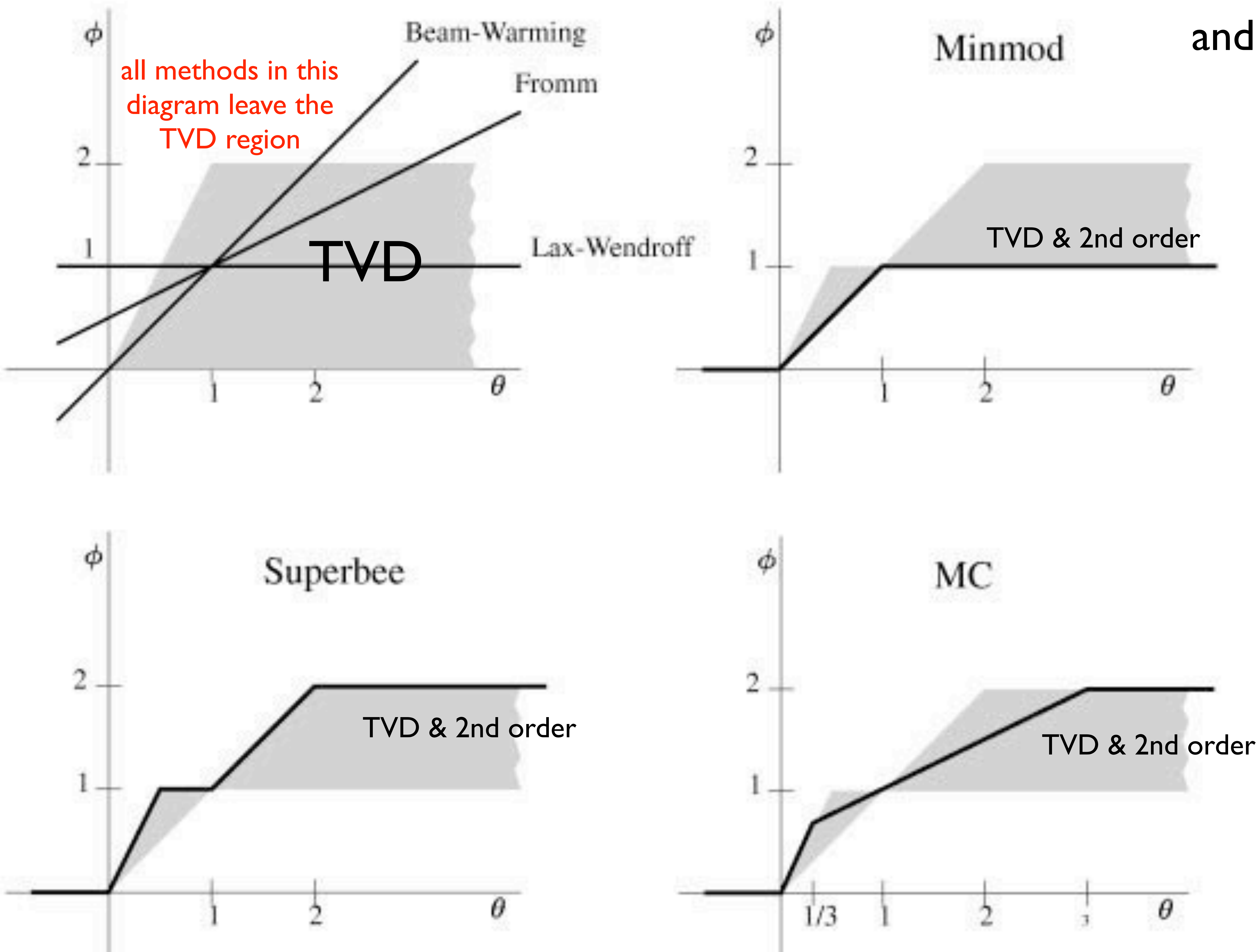
minmod slopes



superbee or MC slopes

Sweby diagram

Regions in which function values $\phi(\theta)$ must lie in order to give TVD and second order TVD methods.

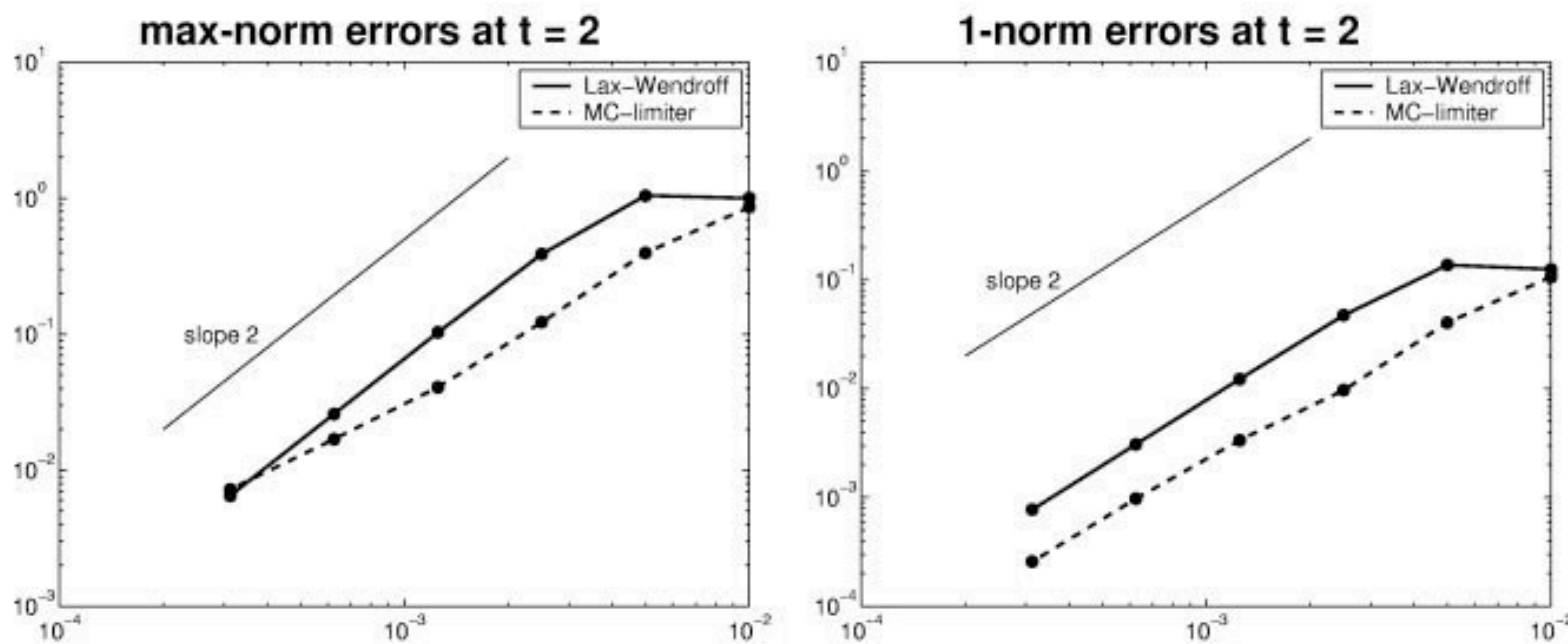


Order of accuracy isn't everything

Comparison of Lax-Wendroff and a high-resolution method on linear advection equation with smooth data.

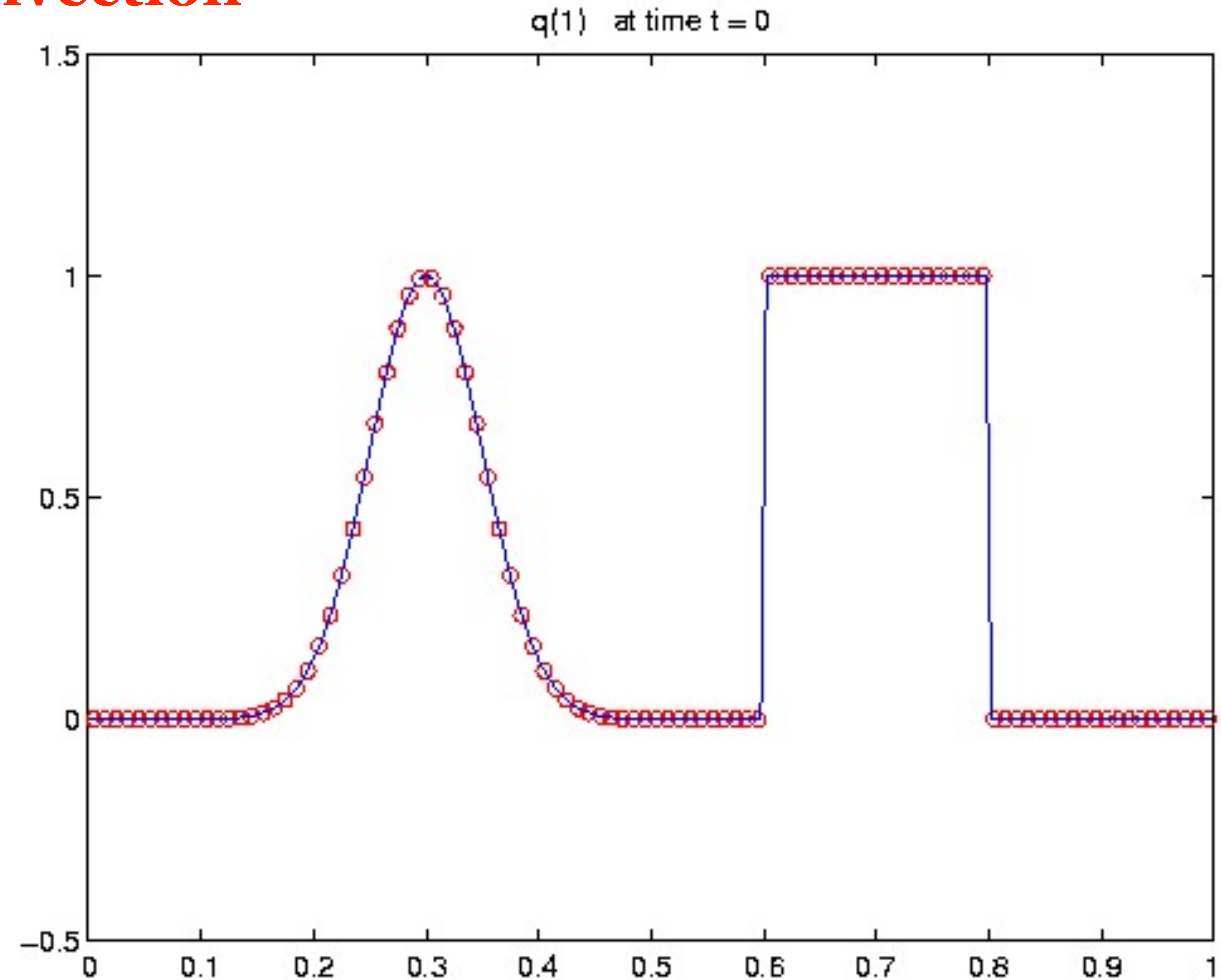
The high-resolution method is not formally second-order accurate, but is more accurate on realistic grids.

Crossover in the max-norm is at 2800 grid points.

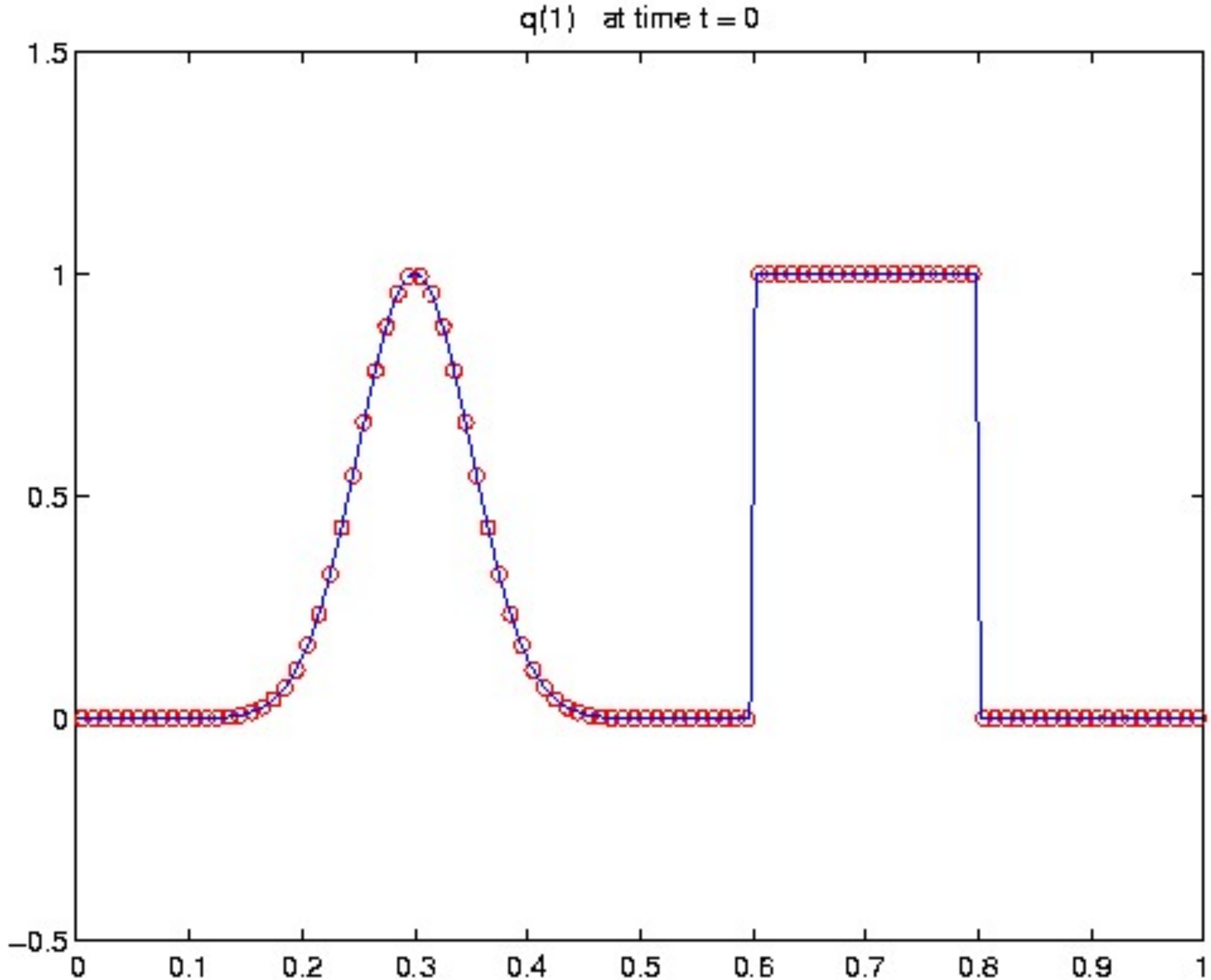


monotone convection

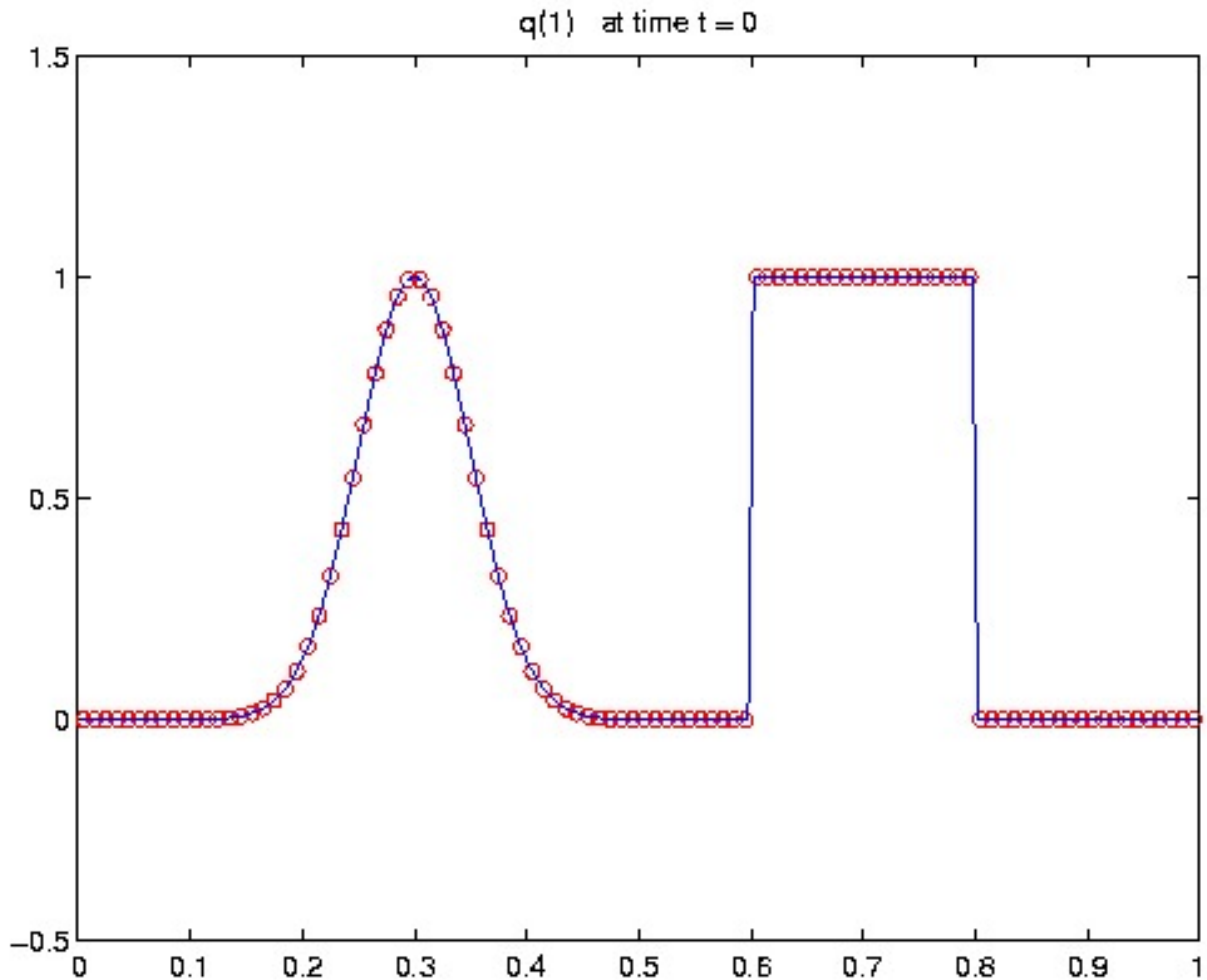
Upwind algorithm



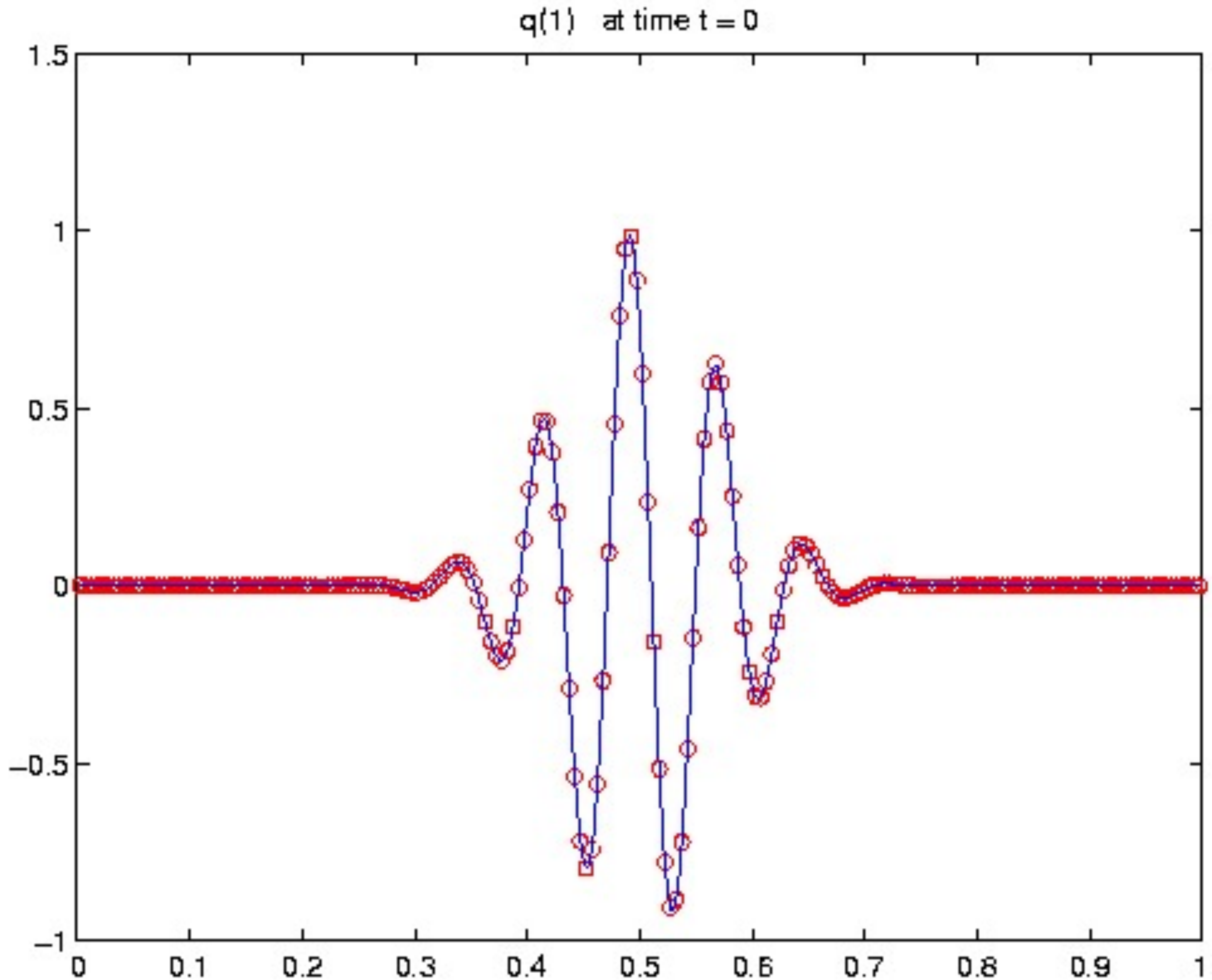
Lax-Wendroff



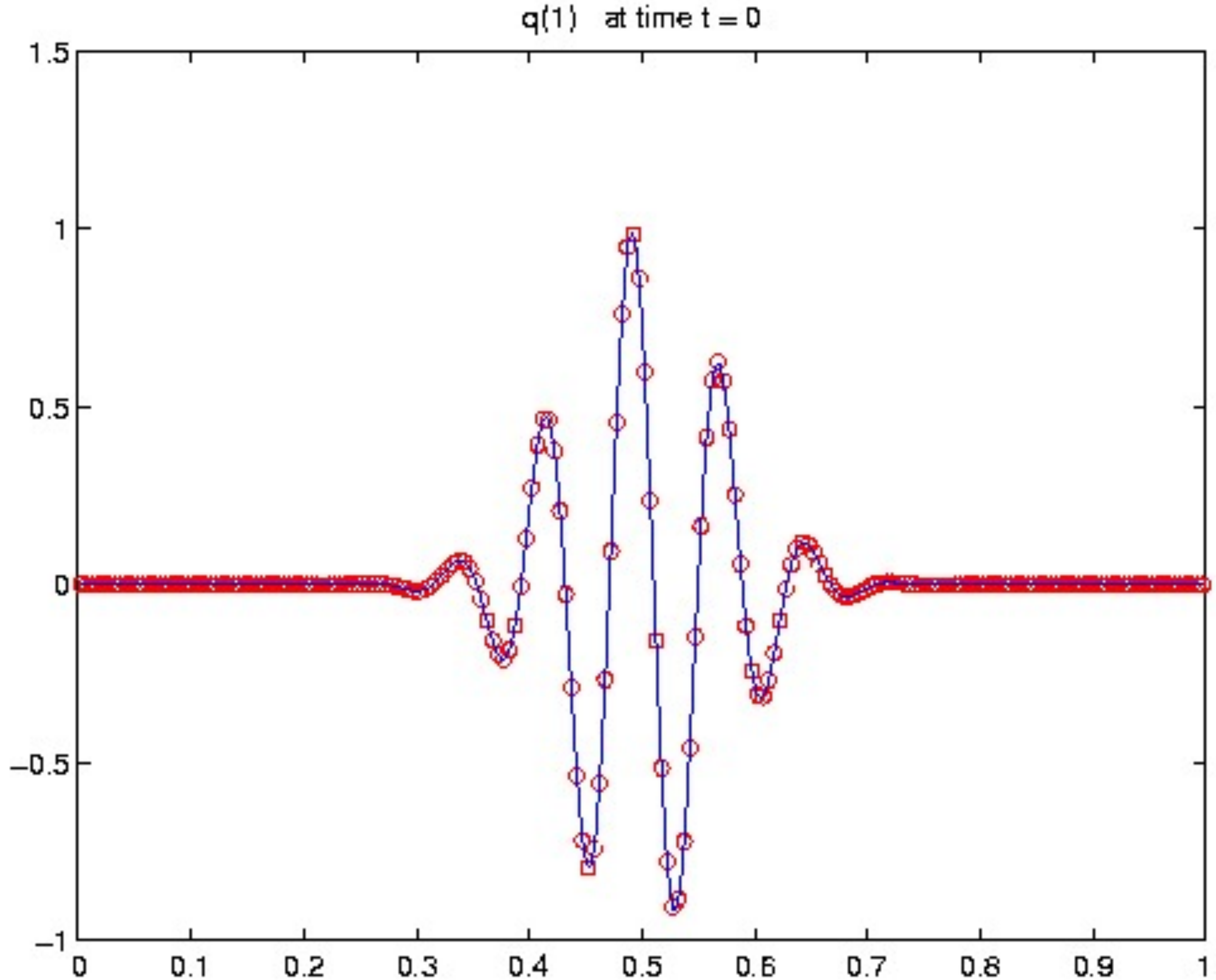
slope limited, MC limiter



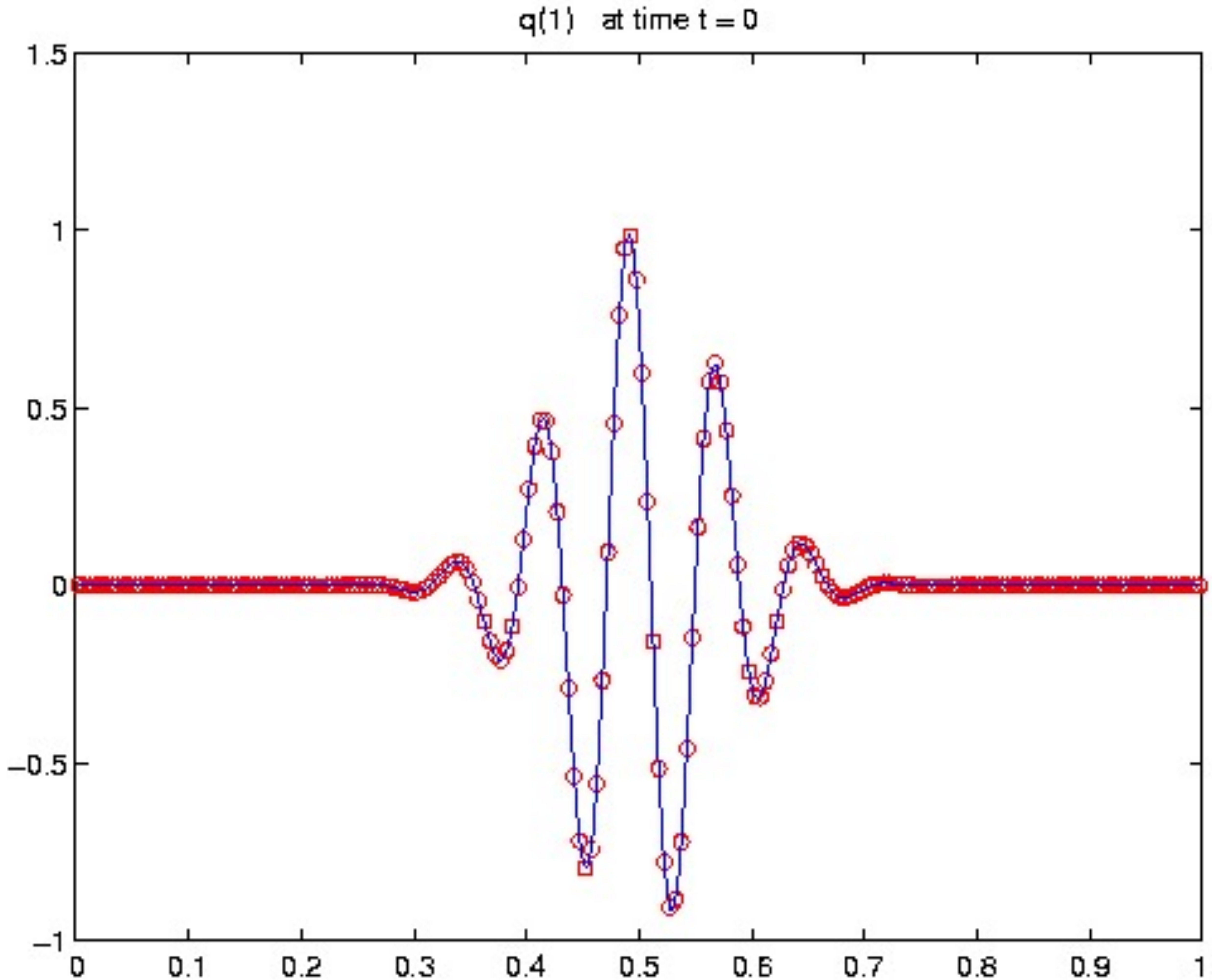
Wave packet: Upwind algorithm



Lax-Wendroff



slope limited, Superbee limiter



Slope limiters and flux limiters

Slope limiter formulation for advection:

$$\tilde{Q}^n(x, t_n) = Q_i^n + \sigma_i^n (x - x_i) \qquad \text{for } x_{i-1/2} \le x < x_{i+1/2}.$$

Applying REA algorithm gives:

$$Q_i^{n+1} = Q_i^n - \frac{ku}{h}(Q_i^n - Q_{i-1}^n) - \frac{1}{2}\frac{ku}{h}(h - uk)(\sigma_i^n - \sigma_{i-1}^n)$$

Flux limiter formulation:

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}(F_{i+1/2}^n - F_{i-1/2}^n)$$

with flux

$$F_{i-1/2}^n = uQ_{i-1}^n + \frac{1}{2}u(h - uk)\sigma_{i-1}^n.$$

Wave limiters

Let $\mathcal{W}_{i-1/2} = Q_i^n - Q_{i-1}^n$.

Upwind: $Q_i^{n+1} = Q_i^n - \frac{ku}{h}\mathcal{W}_{i-1/2}$.

Lax-Wendroff:

$$Q_i^{n+1} = Q_i^n - \frac{ku}{h}\mathcal{W}_{i-1/2} - \frac{k}{h}(\tilde{F}_{i+1/2} - \tilde{F}_{i-1/2})$$

$$\tilde{F}_{i-1/2} = \frac{1}{2}\left(1 - \left|\frac{ku}{h}\right|\right)|u|\mathcal{W}_{i-1/2}$$

High-resolution method:

$$\tilde{F}_{i-1/2} = \frac{1}{2}\left(1 - \left|\frac{ku}{h}\right|\right)|u|\widetilde{\mathcal{W}}_{\mathbf{i-1/2}}$$

where $\widetilde{\mathcal{W}}_{i-1/2} = \Phi_{i-1/2}\mathcal{W}_{i-1/2}$.

Extension to linear systems

Approach 1: Diagonalize the system to

$$v_t + \Lambda v_x = 0$$

Apply scalar algorithm to each component.

Approach 2:

Solve the linear Riemann problem to decompose $Q_i^n - Q_{i-1}^n$ into waves.

Apply a wave limiter to each wave.

These are equivalent.

Nonlinear scalar conservation laws

Burgers' equation: $u_t + \left(\frac{1}{2}u^2\right)_x = 0.$

Quasilinear form: $u_t + uu_x = 0.$

These are equivalent for **smooth** solutions, not for shocks!

Upwind methods for $u > 0.$

Conservative: $U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x}\left(\frac{1}{2}((U_i^n)^2 - (U_{i-1}^n)^2)\right)$

Quasilinear: $U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x}U_i^n(U_i^n - U_{i-1}^n).$

Ok for smooth solutions, not for shocks!

Weak solutions depend on the conservation law

The conservation laws

$$u_t + \left(\frac{1}{2}u^2\right)_x = 0$$

and

$$\left(u^2\right)_t + \left(\frac{2}{3}u^3\right)_x = 0$$

both have the same quasilinear form

$$u_t + uu_x = 0$$

but have different weak solutions,

different shock speeds!

Conservation form

The method

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}(F_{i+1/2}^n - F_{i-1/2}^n)$$

is in **conservation form**.

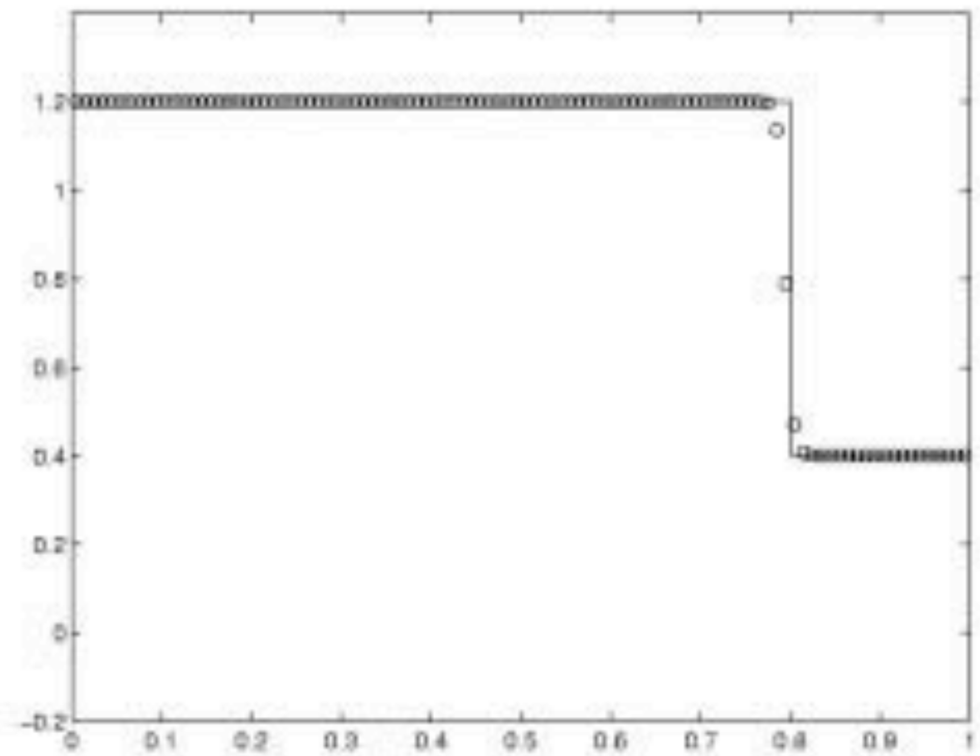
The total mass is conserved up to fluxes at the boundaries:

$$\Delta x \sum_i Q_i^{n+1} = \Delta x \sum_i Q_i^n - \frac{\Delta t}{\Delta x}(F_{+\infty} - F_{-\infty}).$$

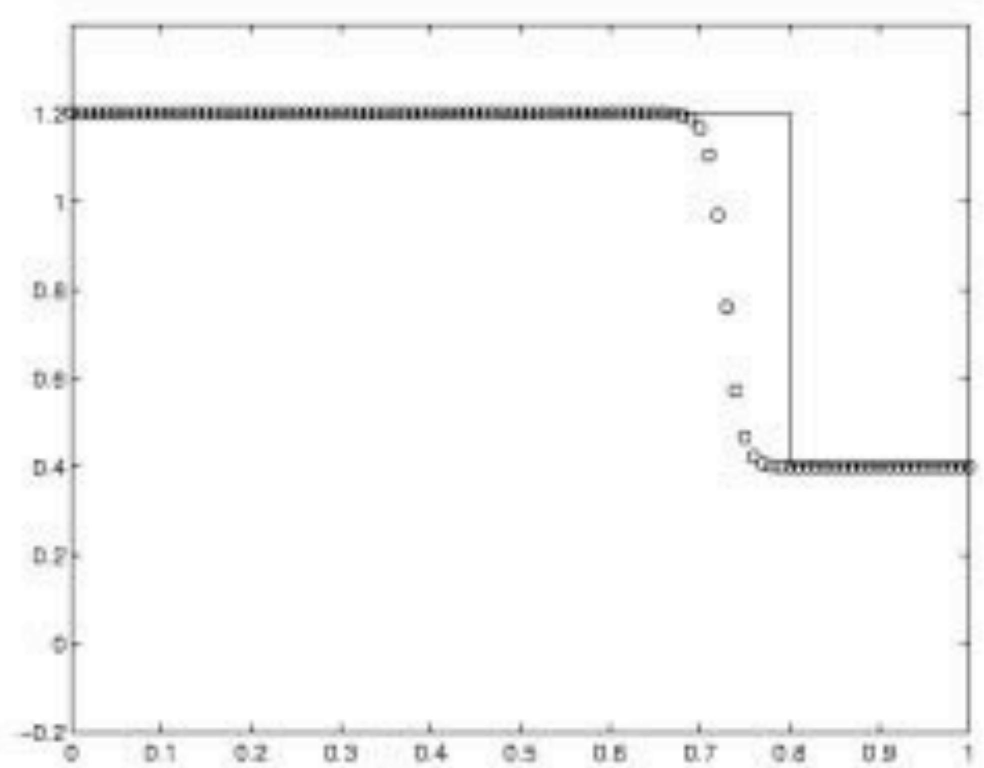
Note: an isolated shock must travel at the right speed!

Importance of conservation form

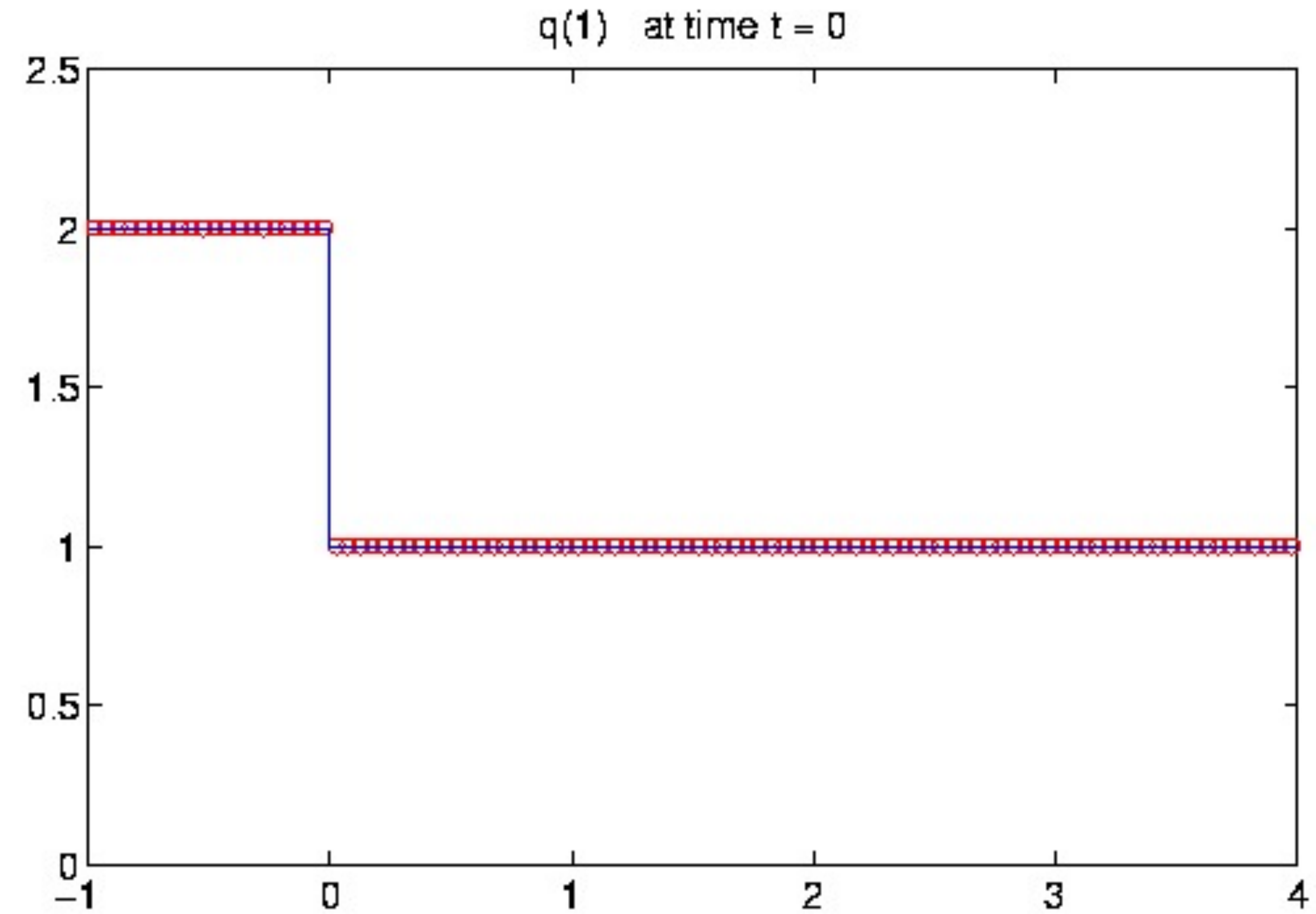
Solution to Burgers' equation using conservative upwind:



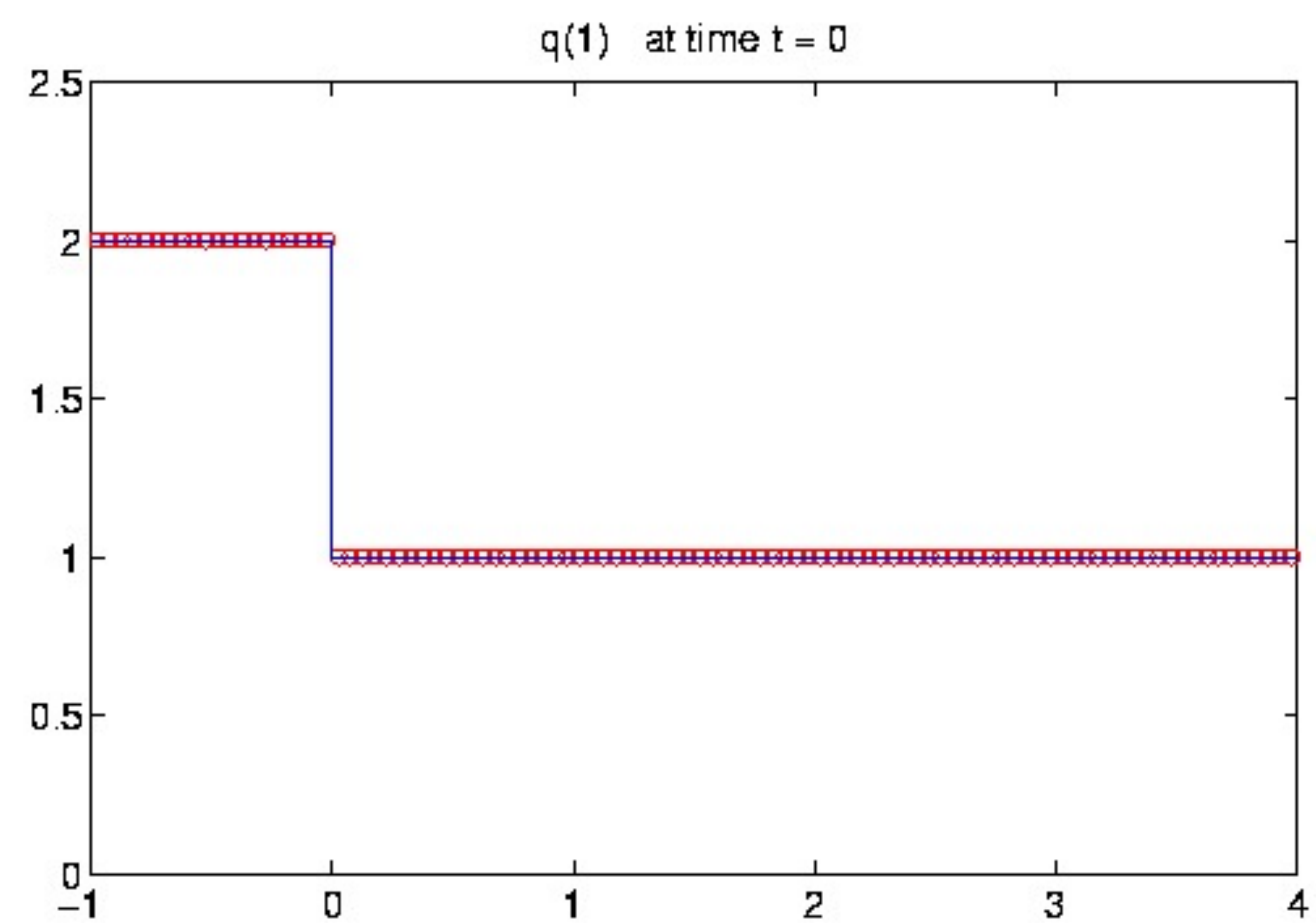
Solution to Burgers' equation using quasilinear upwind:



Burgers' equation solved with an upwind method, to demonstrate that this does not approximate the weak solution properly.



nonconservative method



conservative method

Lax-Wendroff Theorem

Suppose the method is conservative and consistent with
$q_t + f(q)_x = 0$,

$$F_{i-1/2} = \mathcal{F}(Q_{i-1}, Q_i) \quad \text{with } \mathcal{F}(\bar{q}, \bar{q}) = f(\bar{q})$$

and Lipschitz continuity of $\mathcal{F}$.

If a sequence of discrete approximations converge to a function $q(x,t)$ as the grid is refined, then this function is a weak solution of the conservation law.

Note:

Does not guarantee a sequence converges

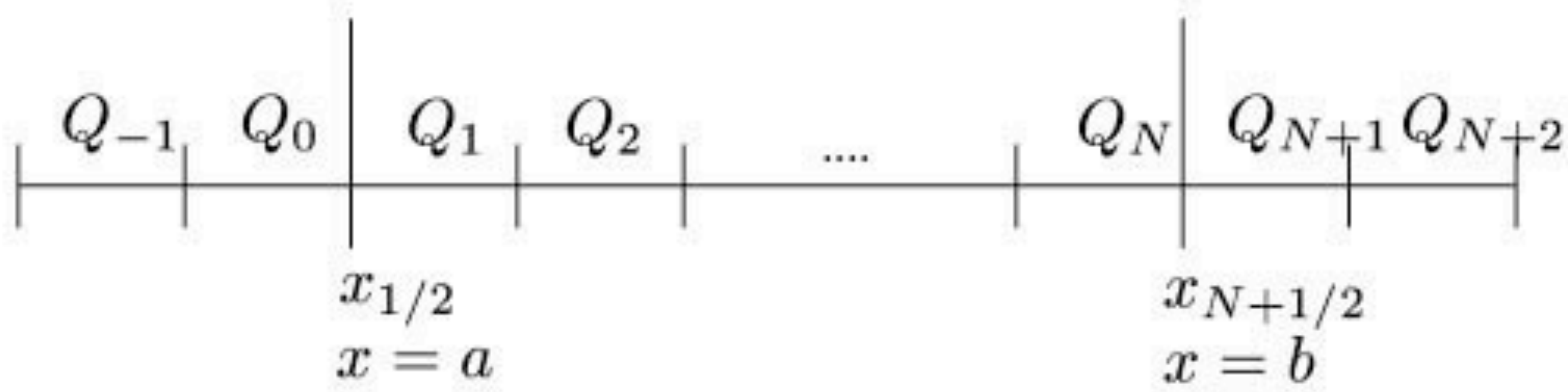
Two sequences might converge to **different** weak solutions.

Also need **stability** and **entropy condition.**

Boundary conditions and ghost cells

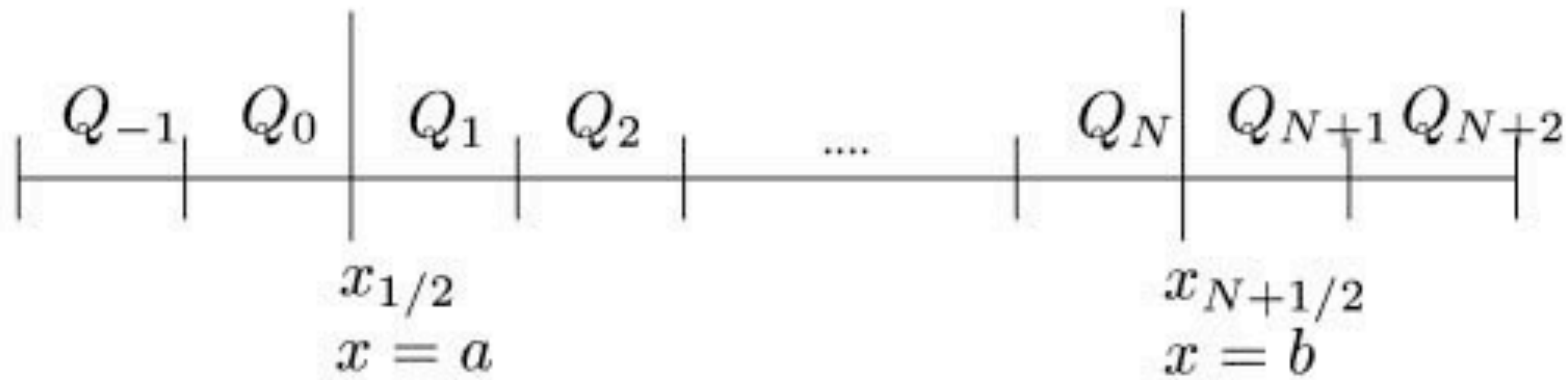
In each time step, the data in cells 1 to N is used to define **ghost cell values** in cells outside the physical domain.

The wave-propagation algorithm is then applied on the expanded computational domain.



The data is extended depending on the physical boundary conditons.

Sample boundary conditions



Periodic:

$$Q^n_{-1} = Q^n_{N-1}, \quad Q^n_0 = Q^n_N, \quad Q^n_{N+1} = Q^n_1, \quad Q^n_{N+2} = Q^n_2$$

Extrapolation (outflow):

$$Q^n_{-1} = Q^n_1, \quad Q^n_0 = Q^n_1, \quad Q^n_{N+1} = Q^n_N, \quad Q^n_{N+2} = Q^n_N$$

Solid wall:

$$\begin{aligned} &\text{For } Q_0 : &\quad p_0 = p_1, &\quad u_0 = -u_1, \\ &\text{For } Q_{-1} : &\quad p_{-1} = p_2, &\quad u_{-1} = -u_2. \end{aligned}$$

Acoustics equations

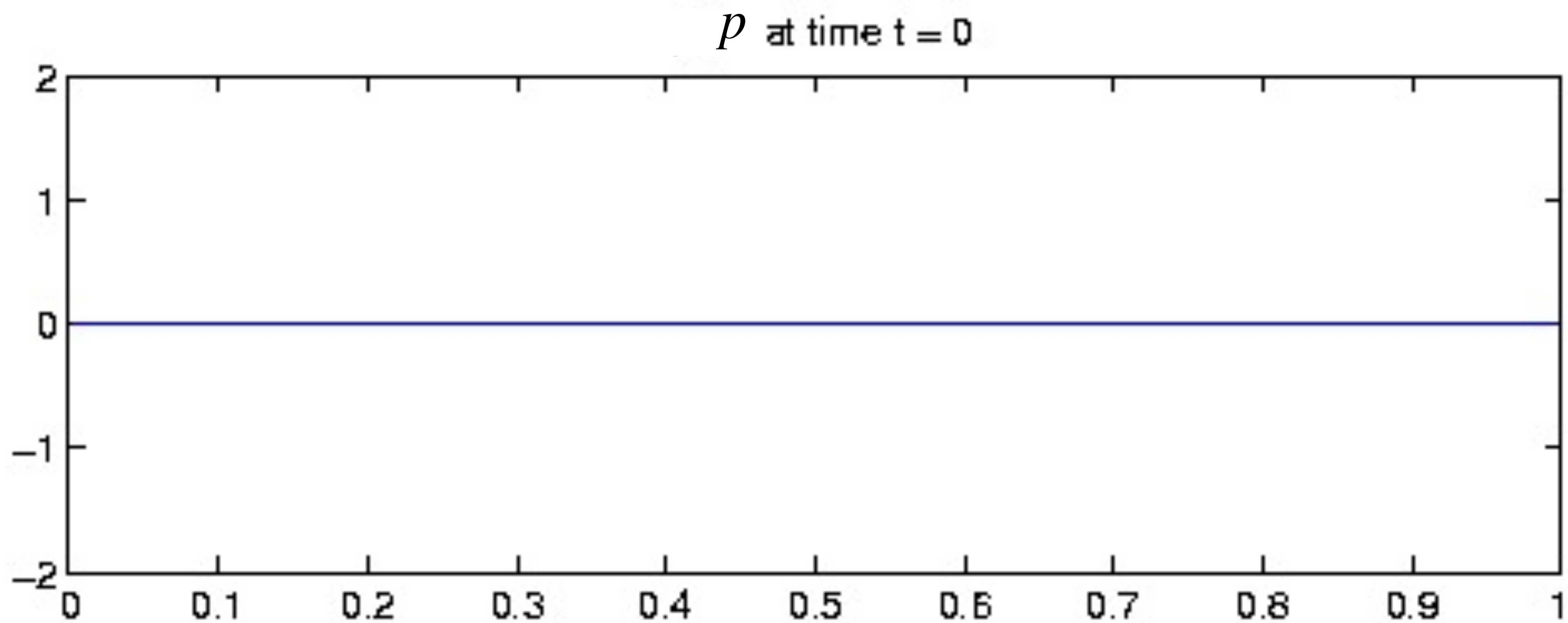
$$\begin{bmatrix} p \\ u \end{bmatrix}_t + \begin{bmatrix} u_0 & K_0 \\ 1/\rho_0 & u_0 \end{bmatrix} \begin{bmatrix} p \\ u \end{bmatrix}_x = 0$$

Computed with 400 grid points and the MC limiter.

Zero initial condition.

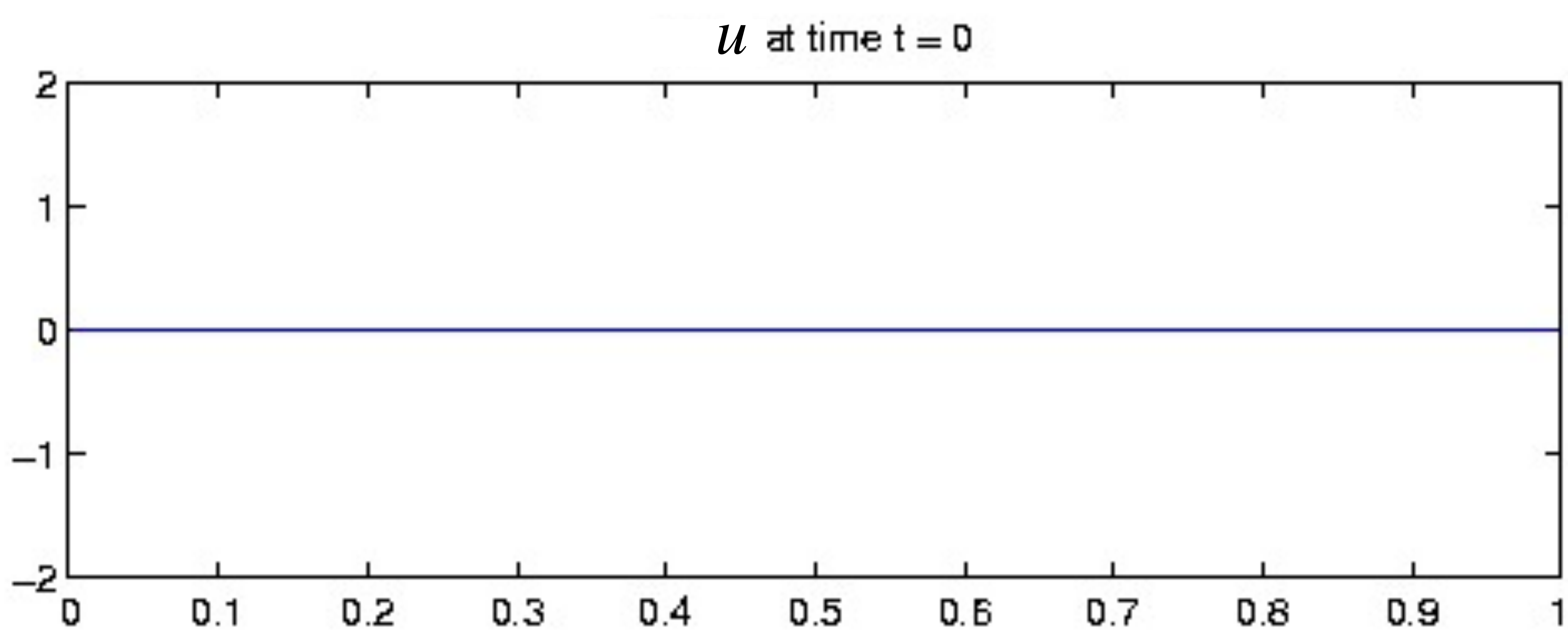
inflow

sinusoidal disturbance enters here



reflecting boundary

inflow



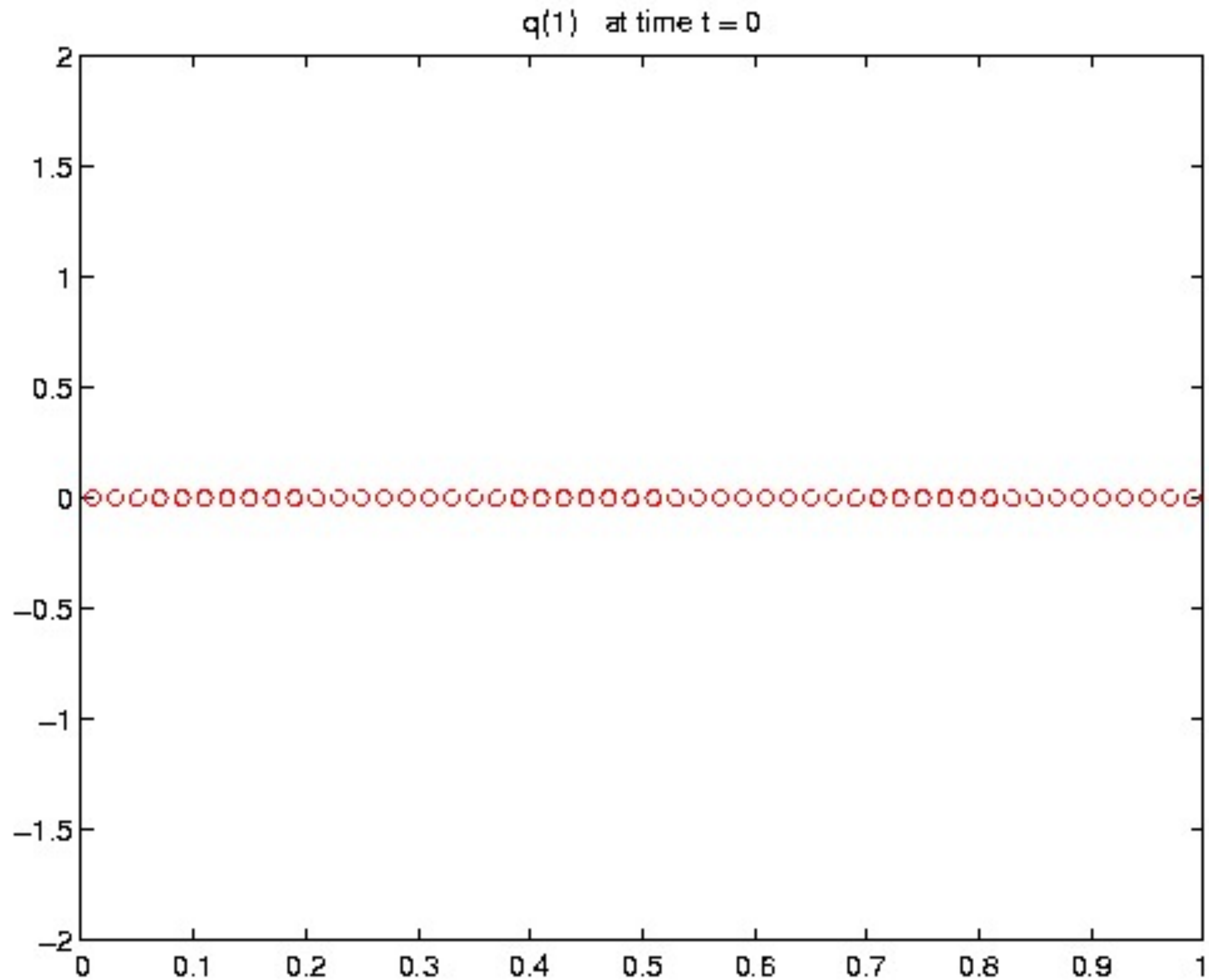
reflecting boundary

Advection equation

Zero initial condition.

inflow

sinusoidal disturbance enters here

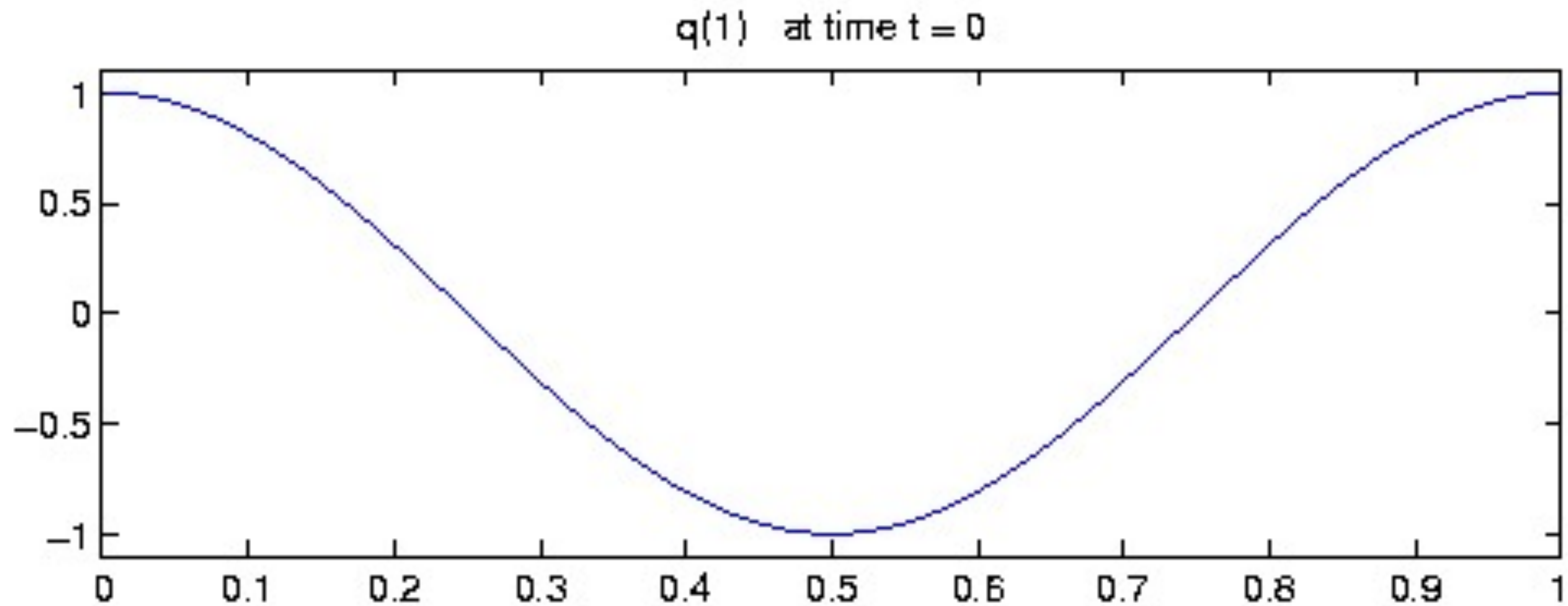


outflow

A standing wave in a closed tube with solid wall boundary conditions.

reflecting boundary

reflecting boundary



reflecting boundary

reflecting boundary