Problem Set 5

Problems to Theoretical Astrophysics, SS 2014

Prof. Dr. Friedrich Röpke, Sebastian Ohlmann
Offices: Campus Hubland Nord, 31.01.017, 31.01.003

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1. Schwarzschild criterion

Show that an alternate criterion for *convective stability* is given by the Schwarzschild criterion,

$$\frac{\mathrm{d}S}{\mathrm{d}z} > 0.$$

Hint: Consider a small fluid element in a star which is heated up and rises adiabatically (i.e. constant entropy in the element); the pressure in the cell is in equilibrium with the outer gas. What can you infer about the gradient of the specific volume in the stellar atmosphere? Convert this into a statement for the entropy stratification.
Hint: Convective stability means that the fluid element is driven back to its initial position; if the stratification is convectively unstable, the fluid element rises further and a large scale motion sets in.

2. Transition to continuum and Fermi energy

a) Assume a single particle in a box of length L. Then the momenta are quantized according to the Schrdinger equation as $\vec{p} = \frac{\hbar\pi}{L}\vec{n}$, where $\vec{n}$ are the quantum number for each dimension. In the limit of large lengths L, discrete sums over energy levels can be transformed into integrals over the continuous momentum,

$$\frac{1}{V}\sum_{\vec{n},\sigma} f(\epsilon_{\vec{n}}) \to g_s \int_0^\infty \mathrm{d}p\; f(\epsilon(p))\nu(p),$$

where $V = L^3$ is the volume, σ is the spin, and g_s the spin degeneracy.

Hint: Include an additional factor $(\Delta n)^3$, where $\Delta n = 1$, and transform it to $(\Delta p)^3$. For large L, this renders the infinitesimal quantity d^3p. Take into account $\vec{n} \geq 0$.

b) For an ideal Fermi gas at $T = 0$, all states are occupied up to the Fermi energy ϵ_F because of the Pauli principle. The number of states equals the number of particles and is given by $N = \sum_{\vec{n},\sigma} n_\mathrm{F}(\epsilon)$, where $n_\mathrm{F}(\epsilon)$ is the Fermi function.

Compute the Fermi momentum p_F and the Fermi energy ϵ_F by assuming a step function for the Fermi function.

Hint: The final result is

$$\epsilon_\mathrm{F} = \frac{\hbar^2}{2m}\left(\frac{6\pi^2 n}{g_s}\right)^{\frac{2}{3}},$$

where $n = N/V$ denotes the number density.

c) Compute the Fermi energy and the Fermi temperature $T_\mathrm{F} = \epsilon_\mathrm{F}/k_\mathrm{B}$ for nuclear matter, a metal, and a white dwarf.

   Hint: For nuclear matter, consider a nucleus obeying the relation $R = 1.25 \times 10^{-15}\,\mathrm{m}\,A^{1/3}$, where A is the total number of nucleons. For the other cases, the density is given by the baryonic density with $n_\mathrm{el} = \rho \frac{Y_\mathrm{e}}{m_\mathrm{b}}$, where Y_e denotes the fraction of electrons to baryons and $m_\mathrm{b} \approx m_\mathrm{u}$ the mean baryonic mass. You can use the following values:

Iron	$\rho = 7.87 \frac{\mathrm{g}}{\mathrm{cm}^3}$	$Y_\mathrm{e} = 0.466$
White dwarf	$\rho = 3 \times 10^6 \frac{\mathrm{g}}{\mathrm{cm}^3}$	$Y_\mathrm{e} = 0.5$

3. Degeneracy in stars

A criterion for degeneracy in the interior of stars can be obtained by comparing the Fermi energy ϵ_F to the mean thermal energy $\frac{3}{2} k_\mathrm{B} T$.

a) Obtain a criterion for $T/\rho^{2/3}$ allowing a statement concerning the degree of degeneracy.

b) Compare the Sun and the white dwarf Sirius B with regard to the degeneracy of electrons in the interior.

   Hint: You can use the following values:

Sun	$T_\mathrm{c} = 1.58 \times 10^7\mathrm{K}$	$\rho_\mathrm{c} = 162 \frac{\mathrm{g}}{\mathrm{cm}^3}$
Sirius B	$T_\mathrm{c} = 7.6 \times 10^7\mathrm{K}$	$\rho_\mathrm{c} = 3 \times 10^6 \frac{\mathrm{g}}{\mathrm{cm}^3}$

4. Pressure for ideal quantum gases

a) Use the grand potential $\Phi = -PV = -\frac{1}{\beta} \ln Z_\mathrm{G}$ to show that

$$P = \frac{g_s}{(2\pi\hbar)^3} \int_0^\infty \mathrm{d}p \, \frac{4\pi p^2}{\exp\left(\beta(\epsilon(p) - \mu)\right) + a} \, \frac{p}{3} \frac{\mathrm{d}\epsilon(p)}{\mathrm{d}p},$$

   where $a = +1$ for fermions and $a = -1$ for bosons.

b) Compute the pressure using the relativistic energy relation $\epsilon(p) = \sqrt{p^2c^2 + m^2c^4} - mc^2$. What do you obtain in the non-relativistic and extreme relativistic limit? Compare to the expression for the energy E and give a relation between pressure and energy for both limits.

5. Polytropes

The equation of state of a degenerate Fermi gas can be written as a polytrope,

$$P = K\rho^{1+\frac{1}{n}}.$$

Compute expressions for the polytrope constant K and the polytrope index n for a completely degenerate Fermi gas

a) in the non-relativistic limit.

b) in the extreme relativistic limit.

6. Equation of state for white dwarfs

The equation of state of white dwarfs can be approximated by an ideal, completely degenerate, *arbitrarily* relativistic Fermi gas.

a) Compute the number density of the electrons as a function of the Fermi momentum p_F.

b) Show that pressure and energy are given by

$$P = \frac{\pi m_\mathrm{e}^4 c^5}{3h^3} f(x),$$
$$e = \frac{\pi m_\mathrm{e}^4 c^5}{h^3} g(x),$$

where

$$f(x) \equiv x(2x^2 - 3)\sqrt{1 + x^2} + 3\,\mathrm{Arsinh}(x),$$
$$g(x) \equiv x(1 + 2x^2)\sqrt{1 + x^2} - \mathrm{Arsinh}(x).$$

Hint: Introduce the parameters

$$z \equiv \frac{p}{m_\mathrm{e} c} \quad \text{and} \quad x \equiv \frac{p_\mathrm{F}}{m_\mathrm{e} c}.$$

c) Although the pressure is dominated by the degenerate electrons, the density is given by the rest mass density of the ions. Compute the density ρ_0 as a function of the mean baryonic rest mass

$$m_\mathrm{B} \equiv \frac{1}{n} \sum_i n_i m_i \qquad \text{(species } i\text{)},$$

the mean number of electrons per baryon Y_e, and the relativity parameter x.

Show that for white dwarf matter consisting of $^{12}\mathrm{C}$ ($m_\mathrm{B} = m_\mathrm{u} = 1.66057 \times 10^{-24}\,\mathrm{g}$) the following relation holds:

$$\rho_0 = 1.9479 \times 10^6 x^3\,\mathrm{g\,cm^{-3}}.$$

Hint: Use $m_\mathrm{B} = m_\mathrm{u} = 1.66057 \times 10^{-24}\,\mathrm{g}$.