# Problem Set 7

**Problems to Theoretical Astrophysics, SS 2014**

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Tutorial on Tuesday, 24.06.2014, 10.15 am

### 1. Gravitational potential

Derive a Poisson equation ($\Delta\Phi = 4\pi G\rho$) for the gravitational potential $\Phi$ from the basic equations of stellar structure,

$$\frac{\mathrm{d}m}{\mathrm{d}r} = 4\pi r^2\rho; \qquad \frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{Gm}{r^2}\rho.$$

### 2. Adiabatic oscillations

In a dynamically stable layer (also called convectively stable), a fluid element is pushed back after some displacement and starts to oscillate around the equilibrium.

a) Assume as equation of state $\rho = \rho(P, T, \mu)$ and write

$$\mathrm{d}\ln\rho = \alpha\mathrm{d}\ln P - \delta\mathrm{d}\ln T + \phi\mathrm{d}\ln\mu,$$

where

$$\alpha = \left(\frac{\partial\ln\rho}{\partial\ln P}\right)_{P,\mu}, \delta = -\left(\frac{\partial\ln\rho}{\partial\ln T}\right)_{T,\mu}, \quad \text{and} \quad \phi = \left(\frac{\partial\ln\rho}{\partial\ln\mu}\right)_{P,T},$$

and $\mu$ is the dimensionless molecular weight (particle mass divided by 1 atomic mass unit).

Use the scale height of pressure $H_P = -\frac{\mathrm{d}r}{\mathrm{d}\ln P}$ and the gradients

$$\nabla = \left(\frac{\partial\ln T}{\partial\ln P}\right)_{\mathrm{s}}, \nabla_{\mathrm{e}} = \left(\frac{\partial\ln T}{\partial\ln P}\right)_{\mathrm{e}}, \quad \text{and} \quad \nabla_\mu = \left(\frac{\partial\ln\mu}{\partial\ln P}\right)_{\mathrm{s}},$$

to rewrite the density difference of the displaced fluid element compared to the surroundings.

*Hint:* The subscript e denotes changes in the element, whereas s denotes changes in the surrounding. Assume the fluid element to be displaced by $\Delta r$.

b) Assume the motion to be adiabatic. Compute the buoyancy force on the fluid element to derive an equation of motion for the diplacement $\Delta r$.

c) Assume harmonic motion and compute the associated frequency, which is called *Brunt-Väisälä* frequency.

d) Use this frequency to derive a criterion for dynamical stability.

### 3. Supermassive stars

For an ideal gas, the pressure is given by

$$P_{\text{gas}} = \frac{\mathcal{R}}{\mu}\rho T,$$

where $\mathcal{R} = 8.31 \times 10^7\,\text{erg}\,\text{K}^{-1}\,\text{g}^{-1}$ denotes the ideal gas constant and $\mu$ the (dimensionless) mean molecular weight, i.e. the average number of atomic mass units per particle.

a) Assume that radiation is present, resulting in an additional pressure term $P_{\text{rad}}$. Show that the equation of state is a polytrope with polytropic index $n = 3$ if the ratio

$$\beta \equiv \frac{P_{\text{gas}}}{P} = \frac{P_{\text{gas}}}{P_{\text{gas}} + P_{\text{rad}}}$$

is assumed to be constant throughout the star.

b) Give the expression for the polytropic constant.

c) Use the expressions for mass and radius of polytropes given in the lecture,

$$M = -4\pi\,\rho_{\text{c}}\,\alpha^3\,x_0^2\,\frac{\mathrm{d}\vartheta}{\mathrm{d}x}\bigg|_{x_0} \qquad \text{and} \qquad R = \alpha x_0,$$

to derive

$$\frac{1-\beta}{\mu^4\beta^4} = 3.0 \times 10^{-3}\left(\frac{M}{M_\odot}\right)^2,$$

originally found by Eddington.

d) How does $\beta$ change with the mass of the star? What are the consequences for very massive stars?

e) Compute $\beta$ for a star with $10^6 M_\odot$ consisting of pure hydrogen.

*Hint:* Use $\mu = 0.5$, as the hydrogen is ionized and the mass of the electrons is negligible.

*Hint:* Use the values

$$x_0 = 6.89685 \qquad \text{and} \qquad -x_0^2\,\frac{\mathrm{d}\vartheta}{\mathrm{d}x}\bigg|_{x_0} = 2.01824$$

for a polytrope with $n = 3$.